Economics 815 **Winter 2015**

Macroeconomic Theory **Thorsten Koeppl**

Answer Key for Assignment 2

Answer to Question 1:

1. The following calculations give the growth rates along a balanced growth path.

   Case 1: Labor-augmenting TFP

$$\begin{aligned}\dot{Y}(t) =& \frac{\partial F\left(K(t), A(t)L(t)\right)}{\partial K(t)}\dot{K}(t) + \frac{\partial F\left(K(t), A(t)L(t)\right)}{\partial A(t)L(t)}L(t)\dot{A}(t) + \\ & \frac{\partial F\left(K(t), A(t)L(t)\right)}{\partial A(t)L(t)}A(t)\dot{L}(t)\end{aligned}$$

   The last term is 0 since labor is constant. Divide both sides by $Y(t)$,

$$\begin{aligned}\frac{\dot{Y(t)}}{Y(t)} &= \frac{K(t)}{Y(t)}\frac{\partial F}{\partial K(t)}\frac{\dot{K}(t)}{K(t)} + \frac{A(t)L(t)}{Y(t)}\frac{\partial F}{\partial A(t)L(t)}\frac{\dot{A}(t)}{A(t)} \\ &\Rightarrow g_Y = \eta_K g_k + \eta_L g_A\end{aligned}$$

   where η_K is the elasticity of output to capital; η_L is the elasticity of output to labor. Along a balanced growth path, it must be that $g_K = g_Y = g$. Hence, we will have

$$g = \frac{\eta_L}{1 - \eta_K} g_A$$

   Case 2: Capital-augmenting TFP

   A similar derivation yields $g_Y = \eta_K(g_A + g_K) \Rightarrow g = \frac{\eta_K}{1-\eta_K} g_A$.

   Case 3: Output-augmenting TFP

   A similar derivation yields $g_Y = g_A + \eta_K g_K \Rightarrow g = \frac{g_A}{1-\eta_K}$

2. When labor grows at rate g_L, we will have

$$\frac{\dot{Y(t)}}{Y(t)} = \frac{K(t)}{Y(t)}\frac{\partial F}{\partial K(t)}\frac{\dot{K}(t)}{K(t)} + \frac{A(t)L(t)}{Y(t)}\frac{\partial F}{\partial A(t)L(t)}\frac{\dot{A}(t)}{A(t)} + \frac{A(t)L(t)}{Y(t)}\frac{\partial F}{\partial A(t)L(t)}\frac{\dot{L}(t)}{L(t)}$$

or

$$g_Y = \eta_K g_k + \eta_L(g_A + g_L).$$

Along a balanced growth path, we will thus have

$$g = \frac{\eta_L}{1-\eta_K}(g_A + g_L).$$

3. Consider first the case with labor-augmenting TFP. A neoclassical production function is given by

$$Y_t = K_t^{\alpha}\left(A_t L_t\right)^{1-\alpha} = A_t^{1-\alpha}K^{\alpha}L_t^{1-\alpha}.$$

Hence, we can define $A_t = B_t^{\frac{1}{1-\alpha}}$ and rewrite the production function as $Y_t = B_t K^{\alpha} L_t^{1-\alpha}$.

A similar approach works for capital-augmenting TFP where we have

$$Y_t = A_t^{\alpha}K^{\alpha}L_t^{1-\alpha}$$

with $A_t = D_t^{\frac{1}{\alpha}}$.

4. It follows immediately from part (b) that Y/L and K/L both grow at rate g. Hence, K/Y is constant.

Next, we show that the wage grows over time. Since we have constant returns to scale, we have for the wage rate that

$$w(t) = (1-\alpha)\frac{Y(t)}{L(t)}$$

and for the interest rate that

$$r(t) = \alpha\frac{Y(t)}{K(t)}.$$

The result now follows immediately from the fact that Y/L grows at a positive rate, while K/Y is constant.

Finally, the factor share of capital needs to be constant which implies immediately that the factor share for labour is also constant. The share of capital is given by

$$s_K(t) = \frac{r(t)K(t)}{Y(t)} = \alpha.$$

Answer to Question 2:

See lecture notes for parts (a) - (d). Differences in calibration will lead to quantitatively different results. Part (e) increases the intertemporal elasticity of consumption, but makes utility linear in labour which corresponds to Hansen (JME, 1985).

Note also that increasing the intertemporal elasticity of substitution *uniformly* across consumption and leisure (e.g. $\gamma = \eta = 5$) will yield a counterintuitive response to labour input in response to a positive technology shock, due to the strong income effect relative to the substitution effect. Try it.

Answer to Question 3:

1. Households take wages w, the interest rate $1+r$, profits Π, as well as the government policy (g, τ_1, τ_2) as given and solve the following problem:

$$\max_{c_1,c_2,n_1,n_2} u(c_1, n_1, c_2, n_2)$$
subject to
$$c_1 + \frac{c_2}{1+r} \leq (1-\tau_1)w_1 n_1 + \frac{(1-\tau_2)w_2 n_2}{1+r} + \Pi$$

The firm takes wages as given and solves

$$\max_{n_t} A n_t^{\alpha} - w_t n_t$$

A *competitive equilibrium* for a government policy (g, τ_1, τ_2) is then given by prices $(w_t, 1+r_t)$ and an allocation (c_1, c_2, n_1, n_2) such that

- households maximize utility taking the policy, profits and prices as given
- firms maximize profits taking wages as given
- markets clear, i.e.

$$\begin{aligned} c_1 + g_1 &= A n_1^{\alpha} \\ c_2 + g_2 &= A n_2^{\alpha} \end{aligned}$$

Note that we have not put any restrictions on government policy. This implies that for values of g that are too high, there might not exist any equilibrium.

2. The government policy is feasible, if it satisfies

$$\begin{aligned} \tau_1 w_1 n_1 &= g_1 \\ \tau_2 w_2 n_2 &= g_2 \\ g_1 + g_2 &= g. \end{aligned}$$

Note that the absence of borrowing and lending does not allow the government to run a (temporary) deficit or surplus. However, by varying τ_1 vs. τ_2 – and, henceforth, g_1 and g_2 – it can shift the tax burden across periods.

3. The firm's problem does not depend on taxes. Profit maximization yields immediately that

$$A\alpha n_t^{\alpha-1} = w_t.$$

For the household, we have the following first-order conditions

$$\frac{(1-n_t)^\eta}{c_t^\gamma} = \frac{\theta}{(1-\tau_t)w_t}.$$

for $t = 1, 2$.

Remark: I did not express the FOCs in terms of the intertemporal Euler equation which is given by

$$\left(\frac{c_2}{c_1}\right)^\gamma = (1+r).$$

This equation is still relevant, but simply pins down the interest rate as a function of taxes τ_1 and τ_2. Since the government cannot borrow or lend and there is a representative household, there cannot be any savings or borrowing. Hence, the problem in each period can be viewed separately – except for the government's constraint of having to raise enough taxes for a total of g.

4. For the computational part, we use the market clearing conditions and the firm's FOCs to eliminate wages and consumption. This yields three equations for (n_1, n_2) and τ_2 – the FOCs of the consumer's problem and the governments budget constraint – which are given by

$$\frac{(1-n_1)^\eta}{[n_1^\alpha(1-\alpha\tau_1)]^\gamma} = \frac{\theta}{(1-\tau_1)\alpha n_1^{\alpha-1}}.$$

where we have already set $A = \theta = 1$.

Furthermore, I first set $\tau_1 = \tau_2 = \frac{g}{2\alpha A n^\alpha}$ and then lower τ_1 as much as possible to finance g in total. The algorithm works as follows. Pick τ_1 and calculate n_1. Then use the FOC for period 2 and the government's budget constraint to jointly solve for τ_2 and n_2. There can potentially be a problem, as for high values of g and very low taxes in the first period there are multiple solutions for τ_2 and n_2. We discuss this further below.

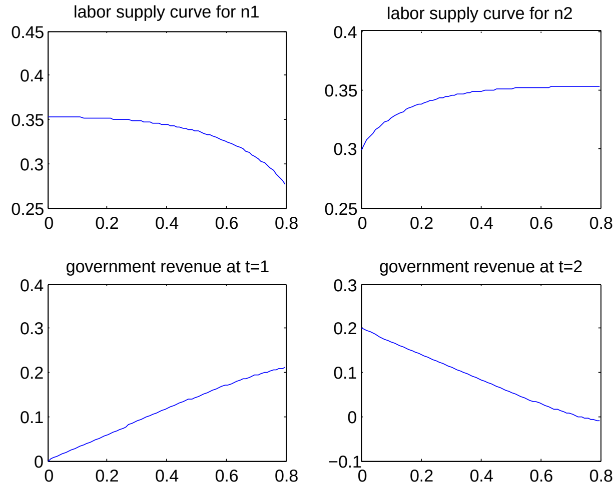

Figure 1: Labor Supply and Govt's Revenue in terms of τ_1

The graphs above show the equilibrium values for labour and government revenue in both periods. Note that we have of course higher values for τ_1 which are not shown in the graph, but we would simply obtain a mirror image in terms of labour supply and government revenue as shown in the pictures.

5. One can easily verify that the welfare maximizing policy given a total expenditure of g across periods is given by $\tau_1 = \tau_2$ or, equivalently $g_1 = g_2$. This is simply a consequence of *tax smoothing* across periods. The intuition is that it is optimal to smooth distortions across time.

6. The graphs below show the equilibrium labour supply and the government revenue as a function of τ for each period. Note that a *Laffer-curve* emerges. First, as tax rate rise, revenue does so as well. Eventually, however, the reduction in labour supply dominates the increase in the tax rate and revenue falls.

   The graph shows that there is a unique tax rate that maximizes government revenue per period. In policy discussions the Laffer-curve has been used to justify that tax cuts will not necessarily reduce tax revenues. However, it is usually not clear for which tax rates this phenomenon occurs. Note that government revenue first changes (almost) linearly with tax rates before falling sharply as taxes approaching 100%. Finally, note that there are multiple solutions for τ and the equilibrium labour supply for all values of

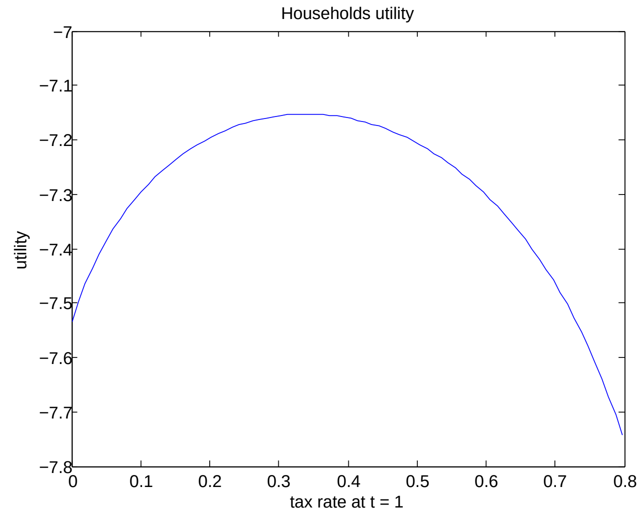

Figure 2: Utility in terms of τ

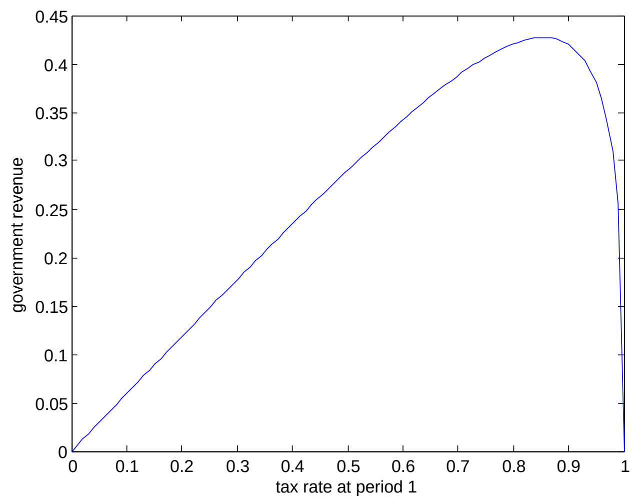

Figure 3: Govt revenue in terms of τ

g. Once we get sufficiently close to the maximum value for g, it might be quite difficult to compute labour supply as we have done above.