Economics 815 **Winter 2014**

Macroeconomic Theory **Thorsten Koeppl**

Answer Key for Assignment 1

Answer to Question 1:

1. The household maximizes utility taking the interest rate and profits as given

$$\max_{c_1,c_2,s} u(c_1,c_2) = \frac{c_1^{1-\gamma}}{1-\gamma} + \beta\frac{c_2^{1-\gamma}}{1-\gamma}$$
$$\text{subject to}$$
$$c_1 + s \leq y$$
$$c_2 \leq rs + \Pi$$

where s denotes savings by the household and Π is the profit from the firm.

Remark: Interpreting savings as investment, we could write here $r + (1-\delta)$ for the return after depreciation. Then, $1+r$ is the gross return on investment. Using $\delta = 1$, we get the above formulation.

2. The firm's problem is to maximize profits taking the interest rate as given

$$f(k) = k^\alpha - rk.$$

3. A competitive equilibrium for this economy is an interest rate r and an allocation (c_1, c_2, s, k) such that

   (a) households maximize utility taking the interest rate and profits as given

   (b) firms maximize profits taking the interest rate as given

   (c) markets clear

$$\begin{aligned} c_1 &= y - k \\ c_2 &= k^\alpha \\ s &= k. \end{aligned}$$

4. From the firm's decision problem, we obtain

$$f'(k) = \alpha k^{\alpha-1} = r.$$

From the consumer's problem we obtain

$$\begin{aligned} c_1^{-\gamma} &= \lambda_1 \\ \beta c_2^{-\gamma} &= \lambda_2 \\ -\lambda_1 + r\lambda_2 &= 0. \end{aligned}$$

This yields the intertemporal Euler equation

$$\left(\frac{c_2}{c_1}\right)^{\gamma} = \beta r.$$

Now we can use the market clearing conditions with $y = 1$ to obtain

$$\left(\frac{k^{\alpha}}{1-k}\right)^{\gamma} = \beta \alpha k^{\alpha-1}.$$

5. Using the parameters of the model we can solve the following non-linear equation for k

$$k^{\alpha\gamma} - (1-k)^{\gamma} \beta \alpha k^{\alpha-1} = 0.$$

The solution is given thus given by

$$\begin{aligned} k^* &= 0.23963 \\ c_1^* &= 0.76037 \\ c_2^* &= 0.65142 \\ r^* &= 0.81553. \end{aligned}$$

6. The graph below shows how the equilibrium values vary with the elasticity of intertemporal substitution γ.

7. The interpretation is straightforward. A higher γ makes people more risk averse, so they would like to smooth consumption more. This also applies to smoothing consumption over time.

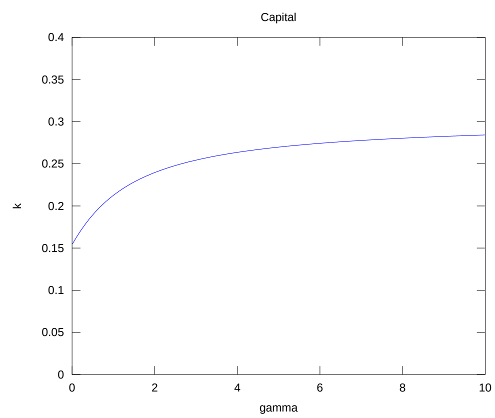

Figure 1: Capital as a function of CRRA coefficient

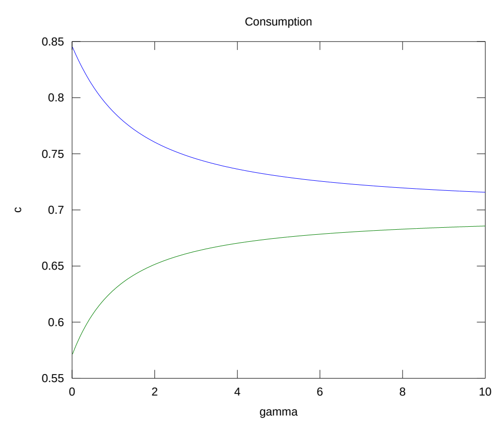

Figure 2: Consumption as a function of CRRA coefficient

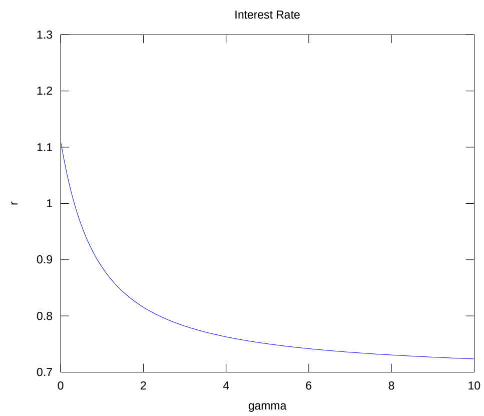

Figure 3: Interest rate as a function of CRRA coefficient

Answer to Question 2:

1. Denote today's probability distribution across states as p_t and tomorrow's probability distribution across states as p_{t+1}. Since today's state is $\underline{y}$, we have $p_t = (1, 0)$.

   We have

   $$p_{t+1} = p_t \Pi.$$

   Hence, the probability of $\underline{y}$ is given by $1 \cdot 0.9 + 0 \cdot 0.1$ and the one of $\bar{y}$ by $1 \cdot 0.1 + 0 \cdot 0.9$, so that $p_{t+1} = (0.9, 0.1)$ which is just the first row of the matrix Π.

2. The long-run stationary distribution solves

   $$p = p\Pi.$$

   This yields two equations in two unknowns given by

   $$\begin{aligned} \underline{p} &= 0.9\underline{p} + 0.1\bar{p} \\ \bar{p} &= 0.1\underline{p} + 0.9\bar{p}. \end{aligned}$$

   Hence, we have that $\underline{p} = \bar{p} = 0.5$.

3. Utility maximization is given by

$$\max_{c_1,\underline{c}_2,\bar{c}_2} u(c_1,c_2) = \frac{c_1^{1-\gamma}}{1-\gamma} + E\left[\beta\frac{c_2^{1-\gamma}}{1-\gamma}\bigg|y_1\right]$$

subject to

$$c_1 + a \leq y_1$$

$$\underline{c}_2 \leq a(1+r) + \underline{y}$$

$$\bar{c}_2 \leq a(1+r) + \bar{y}$$

with the first-order condition given by

$$\begin{aligned} c_1^{-\gamma} &= \lambda_1 \\ \beta\pi(\underline{y}|y_1)\underline{c}_2^{-\gamma} &= \underline{\lambda} \\ \beta\pi(\bar{y}|y_1)\bar{c}_2^{-\gamma} &= \bar{\lambda} \\ -\lambda_1 + (1+r)\underline{\lambda} + (1+r)\bar{\lambda} &= 0. \end{aligned}$$

This yields the intertemporal Euler equation

$$1 = E\left[\beta(1+r)\left(\frac{c_1}{c_2}\right)^\gamma \bigg| y_1\right].$$

Using the endowments, the equilibrium price is given by

$$q = \frac{1}{1+r} = y_1^\gamma\beta\left(\pi(\underline{y}|y_1)\underline{y}_2^{-\gamma} + \pi(\bar{y}|y_1)\bar{y}_2^{-\gamma}\right).$$

Using our parameters, this yields

$$\begin{aligned} \underline{q} &= 0.8325 \\ \bar{q} &= 1.17. \end{aligned}$$

4. The unconditional mean price is given by

$$0.5(\underline{q} + \bar{q}) = 1.00125$$

5. The intertemporal Euler equation is now given by

$$q^e = \beta\left[\pi(\underline{y}|y_1)\left(\frac{c_1}{\underline{c}_2}\right)^\gamma(1+x) + \pi(\bar{y}|y_1)\left(\frac{c_1}{\bar{c}_2}\right)^\gamma(1-x)\right].$$

We also could work with the state prices derived from Arrow-Debreu securities which are given by

$$q(\underline{y}|y_1) = \beta\pi(\underline{y}|y_1)\left(\frac{y_1}{\underline{y}_2}\right)^{\gamma}$$

and

$$q(\bar{y}|y_1) = \beta\pi(\bar{y}|y_1)\left(\frac{y_1}{\bar{y}_2}\right)^{\gamma}$$

respectively.

Using our values, we obtain

$$q^e = 0.81(1 + x) + 0.0225(1 - x) = 0.8325 + 0.7875x.$$

6. It follows immediately that the price for the risky asset is larger than the one for the riskless asset whenever $x > 0$ conditional on the state being $y_1 = \underline{y}$.

   The interpretation is straightforward. The average return on the risky asset is higher given state $y_1 = \underline{y}$. Furthermore, tomorrow there is risk in terms of obtaining higher consumption in state $\bar{y}$. Hence, people would like to hedge against this risk. The risky asset pays more in the low endowment state than in the high endowment state. Hence, the covariance between consumption and asset return is negative, whereas for the risk-free asset it is zero. Hence, the risky asset also provides a hedge against the risky endowment.

   Remark: This is what I did not want for this part of the question. So, let's consider a different asset that pays $1/2 + x$ in state $\underline{y}$ and $1/2 - x$ in state $\bar{y}$. Then, we have again from above

$$q^e = 0.81(1/2 + x) + 0.0225(1/2 - x) = 0.41625 + 0.7875x.$$

   For $x > 0.529$, we have that the price of the risky asset exceeds the price of the risk-free asset (never mind the negative payoff in one state). For this value, however, the average payoff of the risky asset is lower than the one of the risk-free asset, $0.5 + 0.8x$. Why is the price higher? Again, what matters is not only the average return, but also the covariance with consumption which is here negative and, hence, the asset provides a hedge against this risk. If the hedge is good enough – x sufficiently large – its price will hence be higher, as it is a more valuable asset despite the lower average payoff.

7. Conditional on $y_1 = \underline{y}$, we have for the risk-free rate

$$r^f(y_1 = \underline{y}) = \frac{1}{\underline{q}} - 1.$$

For equity, we get for the expected (average) return

$$E[r^e | y_1 = \underline{y}] = 0.9\frac{1+x}{q^e} + 0.1\frac{1-x}{q^e} - 1.$$

The equity premium is thus given by $E[r^e] - r_f$.

Remark: Ideally, we would calculate this premium independent of the state in the first period and take a "long-term" view starting out with the long-run stationary distribution across states. This would involve using unconditional mean returns. Using these, we would have that

$$E[r^e - r^f] = 0.5\left(\frac{0.9(1+x) + 0.1(1-x)}{\underline{q}^e} + \frac{0.1(1+x) + 0.9(1-x)}{\bar{q}^e} - \frac{1}{\underline{q}} - \frac{1}{\bar{q}}\right).$$

Below is the plot of the difference in return between the two assets using the state $\underline{y} = 1$. The risk aversion will influence both q and q^e. Here, the equity premium is negative and becomes larger as γ increases. I admit this is hard for me to interpret.

Answer to Question 3:

Each household i solves the following problem

$$\max_{c_1, c_2} \ln c_1^i + \beta_i \ln c^i$$
$$c_1^i + s^i \leq 1$$
$$c_2^i \leq (1+r)s^i + 1.$$

We can then derive an intertemporal budget constraint that is given by

$$c_1^i + \frac{c_2^i}{1+r} \leq 1 + \frac{1}{1+r}.$$

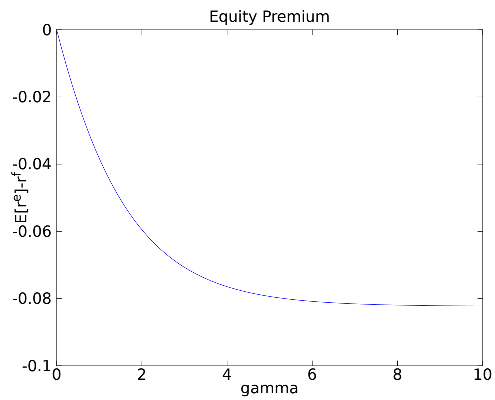

Figure 4: Equity premium as a function of CRRA coefficient

Taking a first-order condition, we obtain for both households

$$\frac{c_2^i}{\beta_i c_1^i} = (1+r).$$

Using this in the budget constraint gives household i's demand for consumption in the first and second period to be

$$\begin{aligned} c_1^i &= \frac{1}{1+\beta_i}\frac{2+r}{1+r} \\ c_2^i &= \frac{\beta_i}{1+\beta_i}(2+r). \end{aligned}$$

Then, we have from market clearing that

$$\sum_i c_t^i = 2$$

for both periods t. Define

$$a = \frac{1}{1+\beta_1} + \frac{1}{1+\beta_2}.$$

Solving we obtain that

$$\begin{aligned} r &= \frac{2a-2}{2-a} \\ 1+r &= \frac{a}{2-a} \\ 2+r &= \frac{2}{2-a}. \end{aligned}$$

Hence, consumption is given by $c_1^1 > c_1^2$ and $c_2^1 < c_2^2$. The household with the higher β (lower discounting) lends to the other household with the higher β (higher discouting).