ECON 815

Analyzing RBC Dynamics

Winter 2015

Dynamics

Dynamics in the economy are given by an endogenous law of motion for capital

$$k_{t+1} = g(k_t, z_t)$$

and the exogenous law of motion for productivity

$$\log z_{t+1} = \rho \log z_t + \epsilon_t.$$

There are many ways to solve for the law of motion of capital. All of these are *approximations* to the actual solution.

We will employ <u>log-linearization</u> to derive such an approximation.

$$\begin{aligned} k_{t+1} &= a_1 k_t + a_2 z_t \\ \lambda_{t+1} &= b_1 k_t + b_2 z_t \\ \log z_{t+1} &= \rho z_t + \epsilon_t \end{aligned}$$

where λ_t is the Lagrange-multiplier which will pin down all “jump” (or decision) variables.

Log-Linearizing

Define the (approximate) deviation from steady state as

$$\hat{x}_t = \log\left(\frac{x_t}{\bar{x}}\right) \approx \frac{x_t - \bar{x}}{\bar{x}}.$$

Approximate a non-linear equation $y_t = f(x_t)$ as a linear equations in terms of percentage deviations from steady state values:

$$\bar{y}e^{\hat{y}_t} = f(\bar{x}e^{\hat{x}_t})$$

A first-order Taylor expansion around $\hat{y}_t = 0$ and $\hat{x}_t = 0$ yields

$$\bar{y}e^0 + \bar{y}e^0(\hat{y}_t - 0) \approx f(\bar{x}e^0) + f'(\bar{x}e^0)\bar{x}e^0(\hat{x}_t - 0)$$

which is

$$\bar{y} + \bar{y}\hat{y}_t \approx f(\bar{x}) + f'(\bar{x})\bar{x}\hat{x}_t$$

and since $\bar{y} = f(\bar{x})$

$$\bar{y}\hat{y}_t = f'(\bar{x})\bar{x}\hat{x}_t.$$

Some Useful Formulas

A general formula is given by

$$y_t = f(x_t, z_t) \Longrightarrow \bar{y}\hat{y}_t = f_x(\bar{x}, \bar{z})\bar{x}\hat{x}_t + f_z(\bar{x}, \bar{z})\bar{z}\hat{z}_t$$

Hence:

$$\begin{aligned}
x_{t+1} = f(x_t) \quad &\Longrightarrow \quad \hat{x}_{t+1} = f'(\bar{x})\hat{x}_t \\
y_t = x_t z_t \quad &\Longrightarrow \quad \hat{y}_t = \hat{x}_t + \hat{z}_t \\
y_t = \frac{x_t}{z_t} \quad &\Longrightarrow \quad \hat{y}_t = \hat{x}_t - \hat{z}_t \\
y_t = x_t + z_t \quad &\Longrightarrow \quad \hat{y}_t = \bar{x}\hat{x}_t + \bar{z}\hat{z}_t \\
y_t = x_t^{\epsilon} \quad &\Longrightarrow \quad \hat{y}_t = \epsilon\hat{x}_t \\
0 = g(x_t, y_t) \quad &\Longrightarrow \quad \hat{y}_t = -\frac{g_x(\bar{x}, \bar{y})\bar{x}}{g_y(\bar{x}, \bar{y})\bar{y}}\hat{x}_t
\end{aligned}$$

Log-linearizing the RBC Model

We can write our model as

$$\begin{aligned}
c_t^{-\gamma} &= \lambda(z_t) \\
\theta(1-n_t)^{-\eta} &= (1-\alpha)\frac{y_t}{n_t}\lambda(z_t) \\
\lambda(z_t) &= \beta E_t[\lambda(z_{t+1})R_{t+1}] \\
R_{t+1} &= \alpha\frac{y_{t+1}}{k_{t+1}} + (1-\delta) \\
c_t + k_{t+1} &= y_t + (1-\delta)k_t \\
y_t &= z_t k_{t+1}^{\alpha} n_t^{1-\alpha} \\
\log z_t &= \rho \log z_{t-1} + \epsilon_t
\end{aligned}$$

In log-linearized form:

$$
\begin{aligned}
-\gamma\hat{c}_t &= \hat{\lambda}_t \\
\hat{n}_t - (1-\eta)\bar{n}\hat{n}_t &= (1-\bar{n})(\hat{\lambda}_t + \hat{y}_t) \\
\hat{\lambda}_t &= E_t[\hat{\lambda}_t + \hat{R}_{t+1}] \\
\bar{R}\hat{R}_{t+1} &= \alpha\frac{\bar{y}}{\bar{k}}\left(\hat{y}_{t+1} - \hat{k}_{t+1}\right) \\
\bar{c}\hat{c}_t + \bar{k}\hat{k}_{t+1} &= \hat{y}\hat{y}_t + (1-\delta)\bar{k}\hat{k}_t \\
\hat{y}_t &= \hat{z}_t + \alpha\hat{k}_t + (1-\alpha)\hat{n}_t \\
\hat{z}_t &= \rho\hat{z}_{t-1} + \epsilon_t
\end{aligned}
$$

where we know the steady state values.

Good news: `DYNARE` can log-linearize equations for us.

Good news: `DYNARE` can solve these equations for us.

The TA will show that we obtain the following linear difference equations for market clearing and the Euler equation

$$
\begin{aligned}
0 &= \psi_1 k_{t+1} + \psi_2 k_t + \psi_3 \lambda_t + \psi_4 z_t \\
0 &= E_t[\psi_5 k_{t+1} + \psi_6 \lambda_{t+1} + \psi_7 \lambda_t + \psi_8 z_{t+1}]
\end{aligned}
$$

where ψ's are just constants.

We can plug in the guess

$$
\begin{aligned}
\hat{k}_{t+1} &= a_1 \hat{k}_t + a_2 \hat{z}_t \\
\hat{\lambda}_{t+1} &= b_1 \hat{k}_t + b_2 \hat{z}_t
\end{aligned}
$$

and the TA will solve for these coefficients.

Interpretation: If $k_t = 0.01$ – a one per cent deviation from steady state – and $z_t = 0$, then k_{t+1} will deviate by $a_1 \cdot 0.01$ per cent.

Impulse Response Functions (IRFs)

We have a linearized law of motion for the state variables and jump variables are just functions of the state.

Consequence: IRFs are non-stochastic and can be calculated by iteration directly.

Procedure:

- start out with steady-state values: $\bar{k}$ and $\log \bar{z} = 0$
- assume a one-standard deviation shock: $\epsilon_0 = \sigma_\epsilon$
- calculate from the law of motion $\{\hat{k}_{t+1}\}_t$ and $\{\hat{\lambda}_t\}_t$
- use these values to calculate series for all other variables

More generally, need to work with a one-time deviation from a sequence of shocks.

Simulations and Comparing Moments

We generate N samples of length T.

- generate M random draws for the productivity shocks
- simulate the linearized economy (all variables of interest) with these shocks
- trim the sample by the first $M - T$ observations
- do this N times

Detrend (if necessary) the simulated data as with the real data.

Then, compute sample moments such as variances, covariances, autocorrelations, etc. from the simulated data as an average across the N samples.

Compare these with the data, including standard errors for the simulated moments.