

# ECON 815

## Uncertainty and Asset Prices

Winter 2015

## Adding Uncertainty

Endowments are now stochastic.

- ▶ endowment in period 1 is known at  $y_t$
- ▶ two states  $s \in \{1, 2\}$  in period 2 with endowment  $y_{s,t+1}$ .
- ▶ there is a probability distribution  $(\pi_1, \pi_2)$

People now maximize expected utility

$$u(c_1) + \beta E[u(c_2)]$$

Key idea:

They face different budget constraints depending on the state.

Consumption in different states is a different good.

## Extending the Framework

With more periods, it is convenient to allow tomorrow's state to depend on today's state.

Example 1: Markov chain with two states

$$\Pi = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}$$

where  $\pi_{ij}$  describes the probability of going from state  $i$  today to state  $j$  tomorrow.

Example 2: AR(1) process

$$y_t = \rho y_{t-1} + \epsilon_t$$

where  $\epsilon_t \sim \mathcal{N}(0, \sigma)$  and  $\rho \in (0, 1)$

We then have that tomorrow's expected values are functions of today's state or

$$E_t[y_{t+1}] = E[y_{t+1} | y_t, \dots]$$

## Decisions under Uncertainty

With two periods, people solve

$$\max E_t[u(c_t) + \beta u(c_{t+1})]$$

subject to

$$c_t + a_t = y_t$$

$$c_{s,t+1} = y_{s,t+1} + (1 + r_{t+1})a_t \text{ for all } s$$

where  $a$  denote savings now and  $r_t$  is a *risk-free* interest rate.

Solution:

$$-E_t[u'(c_t)] + E_t[\beta u'(c_{t+1})(1 + r_{t+1})] = 0$$

or

$$\frac{u'(c_t)}{E_t[\beta u'(c_{t+1})]} = 1 + r_{t+1}$$

## Warning!

- 1) Arithmetic and harmonic means are different.

$$\frac{u'(c_t)}{E_t[\beta u'(c_{t+1})]} \neq E_t \left[ \frac{u'(c_t)}{\beta u'(c_{t+1})} \right]$$

- 2) Jensen's Inequality

$$f'' < 0 \quad \Leftrightarrow \quad E[f(x)] < f(E[x])$$

- 3) Covariance matters

$$E_t[XY] \neq E_t[X]E_t[Y]$$

**End Warning!**

## Asset Prices

What is an asset?

Anything that delivers a payoff in units of consumption across states tomorrow.

Could be a contract, a machine, capital, anything.

More generally, think of a tree.

- ▶ price of the tree today is  $p_t$
- ▶ payoff consist of dividend tomorrow  $d_{t+1}$  ...
- ▶ ... and price of the tree tomorrow  $p_{t+1}$

The return from buying a tree is

$$1 + r(s_{t+1}|s_t) = \frac{d(s_{t+1}|s_t) + p(s_{t+1}|s_t)}{p(s_t)}$$

in state  $s_{t+1}$  tomorrow given today's state is  $s_t$ .

An asset is *risk-free* if it has the same return across all states tomorrow.

Otherwise it is a *risky* asset, with an expected return of

$$E_t[1 + r_{t+1}] = E_t \left[ \frac{d_{t+1} + p_{t+1}}{p_t} \right].$$

We use our model – the intertemporal Euler equation, expectations and asset payoffs – to derive a theory of asset prices  $\{p(s_{t+1}|s_t)\}$ .

## Arrow-Debreu Securities

To do so, we first will price elementary securities called *Arrow-Debreu* securities.

- ▶ tomorrow's states  $s \in \{1, 2, \dots, S\}$
- ▶ today's AD security  $s$  pays exactly one unit of consumption in state  $s$  tomorrow and nothing in any other state or period
- ▶ its price is called the *state price*  $s$
- ▶ think of them as one-period zero coupon bonds

All assets can be thought of as portfolios of AD securities.

Key Idea: If we can price all AD securities, we can price any other security through arbitrage.

This is known as the consumption-based capital asset pricing model (CCAPM) and relies on the notion of *complete markets*.



## Pricing Securities

Suppose there are two states tomorrow and people can only choose AD securities to invest in.

$$\begin{aligned} & \max u(c_t) + \beta E_t[u(c_{t+1})] \\ & \text{subject to} \\ & \quad c_t + q(1|s_t)a(1|s_t) + q(2|s_t)a(2|s_t) \leq y_t \\ & \quad c(s_{t+1}|s_t) \leq y_{t+1} + a(s_{t+1}|s_t) \text{ for } s_{t+1} \in \{1, 2\} \end{aligned}$$

where  $a(s_{t+1}|s_t)$  is the amount of AD security  $s_{t+1} \in \{1, 2\}$  they buy.

Solution:

$$q(s_{t+1}|s_t) = \frac{\beta \pi(s_{t+1}|s_t) u'(c(s_{t+1}|s_t))}{u'(c_t)}$$

where  $\pi(s_{t+1}|s_t)$  is the conditional probability for state  $s_{t+1}$  occurring in period  $t$ .

Example 1: Consider a one-period risk-free bond that pays 1 unit of consumption in each state tomorrow.

Payoffs for the bond are given by:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Hence, its price is equal to a portfolio consisting of one unit of each of the two AD securities. Thus,

$$\begin{aligned} q &= q(1|s_t) + q(2|s_t) = \frac{\beta\pi(1|s_t)u'(c(1|s_t))}{u'(c_t)} + \frac{\beta\pi(2|s_t)u'(c(2|s_t))}{u'(c_t)} \\ &= \beta E_t \left[ \frac{u'(c_{t+1})}{u'(c_t)} \right]. \end{aligned}$$

This implies that the risk-free interest rate  $q = 1/(1 + r^f)$  rate satisfies

$$1 = E_t \left[ \frac{\beta u'(c_{t+1})}{u'(c_t)} \right] (1 + r_{t+1}^f)$$

Example 2: Consider any asset with arbitrary payoff across states equal to  $(x_1, x_2)$ .

Payoffs for this asset are given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Its price must be given by

$$q_x = x_1 q(1|s_t) + x_2 q(2|s_t) = \beta E_t \left[ \frac{u'(c_{t+1})x_{t+1}}{u'(c_t)} \right]$$

Interpret this as equity with payoff  $x_{t+1}$ . We then have that the return on equity satisfies

$$1 = \beta E_t \left[ \frac{u'(c_{t+1})}{u'(c_t)} (1 + r_{t+1}^e) \right]$$

## Consumption Insurance and Risk Premia

We have

$$E[xy] = E[x]E[y] + Cov[xy].$$

This implies for asset pricing that

$$1 = E_t \left[ \beta \frac{u'(c_{t+1})}{u'(c_t)} \right] E_t \left[ \frac{x_{t+1}}{q_x} \right] + \beta Cov \left[ \frac{u'(c_{t+1})}{u'(c_t)}, \frac{x_{t+1}}{q_x} \right]$$

What matters for asset prices?

- ▶ the average payoff and the covariance of payoffs with consumption
- ▶ if negative, it is a hedge which increases the price
- ▶ if positive, people require an additional risk premium which decreases the price