ECON 815

Uncertainty and Asset Prices

Winter 2014

Adding Uncertainty

Endowments are now stochastic.

- endowment in period 1 is fixed at $y_1 = y$
- two states $s \in \{H, L\}$ in period 2, where $y_L < y_H$
- there is a probability distribution (π_L, π_H)

People now maximize expected utility

$$u(c_1) + \beta E[u(c_2)]$$

Key idea:

They face different budget constraints depending on the state.

Consumption in different states is a different good.

Extending the Framework

With more periods, it is convenient to allow tomorrow's state to depend on today's state.

Example 1: Markov chain with two states

$$\Pi = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}$$

where π_{ij} describes the probability of going from state i today to state j tomorrow.

Example 2: AR(1) process

$$y_t = \rho y_{t-1} + \epsilon_t$$

where $\epsilon_t \sim \mathcal{N}(0, \sigma)$ and $\rho \in (0, 1)$

We then have that tomorrow's expected values are functions of today's state or

$$E_t[y_{t+1}] = E[y_{t+1} | y_t, \dots]$$

Decisions under Uncertainty

With two periods, people solve

$$\begin{aligned} & \max E_t[u(c_t) + \beta u(c_{t+1})] \\ & \quad \text{subject to} \\ & \qquad c_t + a_t = y_t \\ & \qquad c_{s,t+1} = y_{s,t+1} + (1 + r_{t+1})a_t \text{ for all } s \end{aligned}$$

where a denote savings now and r_t is a *risk-free* interest rate.

<u>Solution:</u>

$$E_t\left[\frac{u'(c_t)}{\beta u'(c_{t+1})}\right] = 1 + r_{t+1}$$

Warning!

$$E[f(x)] < f(E[x]) \text{ if and only if } f'' < 0$$

which is referred to as Jensen's inequality.

Asset Prices

What is an asset?

- assume again two periods (or equivalently a short-lived asset)
- everything is in units of consumption
- pay price p_t for asset ...
- ... in exchange for payoffs across states tomorrow

Think of a tree. It yields fruit every period (dividend) and can be resold each period. The return from buying a tree is

$$1 + r(s_{t+1}|s_t) = \frac{d(s_{t+1}|s_t) + p(s_{t+1}|s_t)}{p(s_t)}$$

in state s tomorrow.

An asset is *risk-free* if it has the same return across all states tomorrow. Otherwise it is a *risky* asset.

For a risky asset, we have an expected total return of

$$E_t[1 + r_{t+1}] = E_t[\frac{d_{t+1} + p_{t+1}}{p_t}].$$

Note that this return will in general depend on today's state. Why?

- c_t can vary across states
- d might depend on today's state

The key question is then how to determine the asset prices $\{p(s_{t+1}|s_t)\}_t$.

We use our model – the intertemporal Euler equation, expectations and asset payoffs – to derive a theory of asset prices.

Arrow-Debreu Securities

To do so, we first will price elementary securities called *Arrow-Debreu* securities.

- tomorrow's states $s \in \{1, 2, \ldots, S\}$
- today's AD security s pays exactly one unit of consumption in state s tomorrow and nothing in any other state or period
- its price is called the *state price s*
- think of them as one-period zero coupon bonds

All assets can be thought of as portfolios of AD securities.

Key Idea: If we can price all AD securities, we can price any other security through arbitrage.

This is known as the consumption-based capital asset pricing model (CCAPM) and relies on the notion of *complete markets*.

Pricing Securities

Suppose there are two states and people can only choose AD securities to invest in.

$$\begin{aligned}
&\max u(c_t) + \beta E_t[u(c_{t+1})] \\
&\quad \text{subject to} \\
&\qquad c_t + q(1|s_t)a(1|s_t) + q(2|s_t)a(2|s_t) \leq y_t \\
&\qquad c(s_{t+1}|s_t) \leq y_{t+1} + a(s_{t+1}|s_t) \text{ for all } s_{t+1}
\end{aligned}$$

where $a(s_{t+1}|s_t)$ is the amount of AD security $s_{t+1}|s_t$ they buy.

<u>Solution:</u>

$$q(s_{t+1}|s_t) = \frac{\beta\pi(s_{t+1}|s_t)u'(c(s_{t+1}|s_t)}{u'(c_t)}$$

where $\pi(s_{t+1}|s_t)$ is the conditional probability for state s_{t+1} occurring in period t.

Example 1: Consider a one-period risk-free bond that pays 1 unit of consumption in each state tomorrow.

Payoffs for the bond are given by:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Hence, its price is equal to a portfolio consisting of one unit of each of the two AD securities. Thus,

$$q = q(1|s_t) + q(2|s_t) = \frac{\beta\pi(1|s_t)u'(c(1|s_t))}{u'(c_t)} + \frac{\beta\pi(2|s_t)u'(c(2|s_t))}{u'(c_t)}$$

$$= \beta E_t\left[\frac{u'(c_{t+1})}{u'(c_t)}\right].$$

This implies that the risk-free interest $q = 1/(1 + r^f)$ rate solves the equation

$$1 = E_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)}\right](1 + r^f_{t+1})$$

Example 2: Consider any asset with arbitrary payoff across states equal to (x_1, x_2).

It's price must be given by

$$q_x = x_1 q_{1,t} + x_2 q_{2,t} = \beta E_t \left[\frac{u'(c_{t+1}) x_{t+1}}{u'(c_t)} \right]$$

Interpret this as equity with payoff $x_{t+1} = d_{t+1} + q_{t+1}$. We then have that the return on equity is defined by

$$1 = \beta E_t \left[\frac{u'(c_{t+1})}{u'(c_t)} (1 + r^e_{t+1}) \right]$$

Careful! The return on equity is also a random variable.

Consumption Insurance and Risk Premia

We have

$$E[xy] = E[x]E[y] + Cov[xy].$$

This implies for asset pricing that

$$q_x = E_t\left[\beta \frac{u'(c_{t+1})}{u'(c_t)}\right] E_t[x] + \beta Cov[\frac{u'(c_{t+1})}{u'(c_t)}, x]$$

What matters for asset prices?

- the average payoff ...
- ... and the covariance of payoffs with consumption
- if negative, it is a hedge which increases the price
- if positive, people require an additional risk premium which decrease the price