Lecture Notes

Public vs. Private Information

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1 Heterogeneous Beliefs Matter

1.1 Perfect Info and Multiple Equilibria

- state θ is drawn from $\mathcal{U} \sim [0,1]$
- action for investor $i \in [0,1]$: attack currency or not $\pi \in \{0,1\}$
- action by large player: defend or not after observing investors' decisions

Payoffs of defending for large player

$$v - c(\alpha, \theta) \tag{1.1}$$

where α is the # of investors attacking and $\partial c/\partial\theta < 0$, $\partial c/\partial\alpha > 0$.

Payoffs of attack for investors:

$$e - f(\theta) - t \text{ if attacks succeed} \qquad - t \text{ if attack fails} \tag{1.2}$$

with $\partial f/\partial\theta > 0$

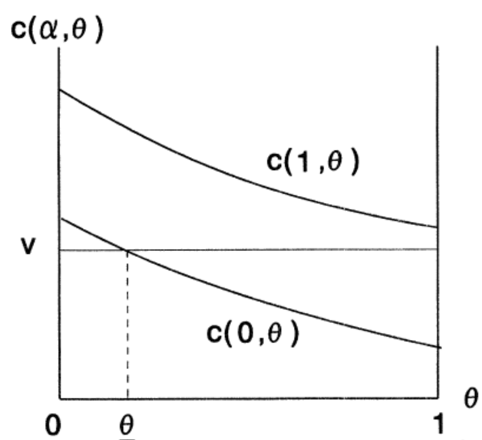

FIGURE 1. COST AND BENEFIT TO THE GOVERNMENT IN MAINTAINING THE CURRENCY PEG

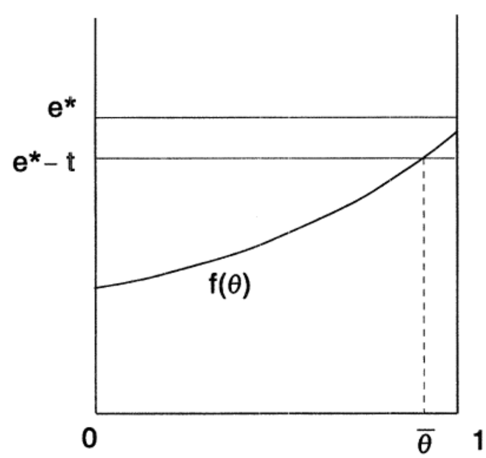

FIGURE 2. THE MANAGED EXCHANGE RATE AND THE EXCHANGE RATE IN THE ABSENCE OF INTERVENTION AS A FUNCTION OF THE STATE OF FUNDAMENTALS

There are three equilibrium regions:

1. $[0, \underline{\theta}]$ – attack and no defense
2. $[\underline{\theta}, \bar{\theta}]$ – attack and no defense & no attack and defense
3. $[\bar{\theta}, 1]$ – no attack a dominant strategy

1.2 A Simple Game of Imperfect Information

Now we change the game slightly. Large player observes θ, but investors observes θ only with noise. Also, set $f(\theta) = F$ and $c(a, \theta) = a - \theta$.

Assumption: Investor i observes private signal x_i which is drawn from $\mathcal{U} \sim [\theta - \epsilon, \theta + \epsilon]$

How do we find the now[1] unique equilibrium?

Step 1: What is the payoff for attacking?

An attack of size α is successful if and only if $c(\alpha, \theta) = \alpha - \theta > v$.

Then, investor attacks if and only if

$$E[\mathbf{1}_{\{c(\alpha,\theta)>v\}}(e - F)|x_i] - t = (e - F)E[\mathbf{1}_{\{\alpha>v+\theta\}}|x_i] - t > 0 \tag{1.3}$$

Key issue: this depends on x_i and on α which depends on *all other signals.*

Step 2: Guess the cut-off rule $\alpha = \mathcal{P}[x_i < k]$ for investors to attack.

We have[2]

$$\mathcal{P}[x_i < k] = \int_{\theta-\epsilon}^{k} dF(x_i) = \frac{1}{2\epsilon}(k - \theta + \epsilon)$$

[1] Proof of uniqueness is hard and the idea is given in Morris and Shin, AER (1998).

[2] Note that this probability could be 1 or 0 depending on the true θ and the error ϵ. We look here at cases where the true θ is sufficiently close to the cut-off k.

Thus, attack is successful if and only if

$$\frac{1}{2} + \frac{k-\theta}{\epsilon} = \alpha(\theta) \geq v + \theta \tag{1.4}$$

Hence, there will be some θ^* such that there is an attack if and only if $\theta \leq \theta^*$.

<u>Step 3:</u> Determine k, the individual cut-off level for an attack.

Consider the marginal investor. He attacks when $x_i = k$, knowing that an attack takes place if and only if $\theta \leq \theta^*$.

His indifference condition is given by[3]

$$\begin{aligned} t &= \int_{\{\theta|\alpha \geq a(\theta)\} \cap [k-\epsilon, k+\epsilon]} (e-F) dF(\theta) \\ &= (e-F)\frac{1}{2\epsilon} \int_{\{\theta|\theta \leq \theta^*\} \cap [k-\epsilon, k+\epsilon]} d\theta \\ &= (e-F)\frac{1}{2\epsilon} \int_{k-\epsilon}^{\theta^*} d\theta \\ &= \frac{(e-F)}{2\epsilon}(\theta^* - k + \epsilon). \end{aligned}$$

<u>Conclusion:</u> We have two equations in the two unknowns (k, θ^*). An attack takes place if and only if

$$\theta \leq \theta^* = 1 - v - \frac{t}{e-F} \tag{1.5}$$

and individual investors attack if and only if

$$x_i \leq \theta^* + \epsilon \left(1 - \frac{2t}{e-F}\right), \tag{1.6}$$

so that not everyone necessarily attacks if θ is sufficiently close to θ^*.

<u>What's going on here?</u>

Investors cannot say anymore that it is common knowledge that the state is θ based on what their signal says about the true state and what other investors can possibly believe about what they know about the state.

[3]Again, this condition holds only with equality if θ is sufficiently close to θ^*.

Observing a signal $x \geq \tilde{\theta} + \epsilon$ rules out states such that $\theta \leq \tilde{\theta}$.

But one needs to observe a signal $\tilde{\theta} + 3\epsilon$ to know that everyone knows that the state cannot be $\theta \leq \tilde{\theta}$.

After n iterations that says that everyone knows that everyone knows ... (n times) that the state is at least $\tilde{\theta}$ if you observe a signal $\tilde{\theta} + (2n-1)\epsilon$. Hence, it is never common knowledge ($n \to \infty$) that $\theta \leq \tilde{\theta}$.

So, even if we know that the fundamental would not give rise to an attack, we attack nonetheless, as others could believe that others believe and so on that the state is really bad. Based on these higher order beliefs there will be an attack even if the underlying fundamental would not give rise to an attack with common knowledge. The key that this works here is that – even though the signals are private – all investors still have common knowledge *about the optimal strategies* of others.

2 Adding Public to Private Info

2.1 A Simple Coordination Game with Imperfect Information

- nature draws a state θ from an uninformative prior (uniform distribution on the real line)
- agents receive a *public* signal $y = \theta + \eta$, where $\eta \sim \mathcal{N}(0, 1/\alpha)$
- agents receive a *private* signal $x_i = \theta + \epsilon_i$, where $\epsilon_i \sim \mathcal{N}(0, 1/\beta)$
- agents take an action $a_i \in I\!R$.

Pay-off function:

$$u_i(\mathbf{a}, \theta) = -(1-r)(a_i - \theta)^2 - r(a_i - \bar{a})^2 + r \int_0^1 (a_j - \bar{a})^2 dj \tag{2.1}$$

The second part of the payoff function makes actions strategic complements (substitutes) if

$r > 0$ ($r < 0$). We set $r \in (0, 1)$. Hence, every agent tries to guess what other agents will do given his own and the public signal.

The third part of the payoff function specifies that this pay-off externality is zero sum (nets out across agents). Also, it does not directly affect the agents decisions. We use a normalized social welfare function

$$W(\mathbf{a}, \theta) = \frac{1}{1-r} \int_0^1 u_i(\mathbf{a}, \theta) di = - \int_0^1 (a_i - \theta)^2 di \tag{2.2}$$

Key idea: A public signal is useful to get a_i closer to θ (informational role), but it is also socially harmful if it coordinates people away from their own signals (focal point for beliefs).

2.2 Benchmark

The agent's first-order condition is given by

$$a_i = (1-r)E_i(\theta) + rE_i(\bar{a}) \tag{2.3}$$

Thus, the optimal action is a weighted average of the expected fundamental and the expected average decision.

Perfect Information: Suppose θ is common knowledge. Then, $a_i^* = \theta$ for all i. To see this just integrate over a_i^*.

Only Public Signal: Suppose everyone observes the signal y. We have that

$$a_i(y) = (1-r)E(\theta|y) + r \int_0^1 E(a_j|y) dj = (1-r)E(\theta|y) + r\bar{a} \tag{2.4}$$

since all actions a_j are measurable w.r.t. y. Hence again integrating, we get that $a_i^* = y$ for all i.

How should we set the precision of the public signal?

We want to maximize expected welfare conditional on any θ that was drawn by nature:

$$E(W|\theta) = -E[(y-\theta)^2|\theta] = -\frac{1}{\alpha} \tag{2.5}$$

Hence, $\alpha = \infty$ or infinitely precise signal.

2.3 The Role of Private Signals

With private and public signals, agents form the posterior belief

$$E_i(\theta) = \frac{\alpha y + \beta x_i}{\alpha + \beta} \tag{2.6}$$

Hence, the belief is a weighted average of the two signals, where the weights correspond to the precision of the signals.

We guess and verify the equilibrium.

Step 1: We guess a linear decision rule $a_j = \kappa x_j + (1-\kappa)y$.

Then we have for individual beliefs about other people's actions[4]

$$\begin{aligned} E_i(\bar{a}) = \int_0^1 E_i(a_j)dj &= \kappa \int_0^1 E(x_j|x_i, y)dj + (1-\kappa)y \\ &= \kappa \int_0^1 E(\theta + \epsilon_j|x_i, y)dj + (1-\kappa)y \\ &= \kappa E(\theta|x_i, y) + (1-\kappa)y \\ &= \kappa \left(\frac{\alpha y + \beta x_i}{\alpha + \beta}\right) + (1-\kappa)y \\ &= \left(\frac{\kappa\beta}{\alpha+\beta}\right) x_i + \left(1 - \frac{\kappa\beta}{\alpha+\beta}\right) y \end{aligned}$$

Step 2: Use the FOC.

We have that $a_i = (1-r)E_i(\theta) + rE_i(\bar{a})$. Using the above results, we obtain that

$$\kappa = \frac{\beta(1-r)}{\alpha\beta(1-r)} \tag{2.7}$$

[4] Note that we integrate with respect to the information set x_i and y to obtain person i's expectation.

which gives us comparing with the guess

$$a_i^* = \frac{\alpha y + \beta(1-r)x_i}{\alpha + \beta(1-r)} \tag{2.8}$$

Conclusion: The public signal *reduces* the weight on the private signal to the extent of the strategic complementarity r.

2.4 Intuition for the Result

The solution method above is not very instructive. Hence, we use brute force to derive the equilibrium.

The only source of heterogeneity among agents is their private signal x_j. Denote the average expectation over θ by $\bar{E}$.

From the best response function, we got

$$\begin{aligned} a_i &= (1-r)E_i(\theta) + rE_i(\bar{a}) \\ &= (1-r)E_i(\theta) + rE_i((1-r)\bar{E}(\theta) + r\int E_j(\bar{a})dj) \\ &= (1-r)E_i(\theta) + (1-r)rE_i(\bar{E}(\theta)) + (1-r)r^2E_i(\bar{E}^2(\theta)) + \ldots \\ &= (1-r)\sum_{k=0}^{\infty} r^k E_i(\bar{E}^k(\theta)) \end{aligned}$$

What is $E_i(\bar{E}(\theta))$?

We got

$$\bar{E}(\theta) = \int_0^1 E_j(\theta)dj = \int_0^1 \frac{\alpha y + \beta x_j}{\alpha + \beta} dj = \frac{\alpha + \beta\theta}{\alpha + \beta} \tag{2.9}$$

which yields for agent i's 2nd order expectations

$$E_i(\bar{E}(\theta)) = E\left(\frac{\alpha y + \beta\theta}{\alpha + \beta}|x_i, y\right) = \frac{\alpha y + \beta\frac{\alpha y + \beta x_i}{\alpha + \beta}}{\alpha + \beta} \neq E_i(\theta) \tag{2.10}$$

and similarly for the n-th order expectation.

2.5 Is Public Information "Good"?

The social welfare optimum is given by the case $r = 0$. Hence, the social optimum weighs the signals according to their precision so that

$$a_i = \theta + \frac{\alpha\eta + \beta\epsilon}{\alpha + \beta} \tag{2.11}$$

But in equilibrium with $r \in [0, 1]$ we have

$$a_i = \theta + \frac{\alpha\eta + \beta(1-r)\epsilon}{\alpha + \beta(1-r)} \tag{2.12}$$

The expected welfare given θ is

$$E[W(\mathbf{a}, \theta)|\theta] = -E[\int_0^1 (a_i - \theta)^2|\theta] = -\frac{\alpha + \beta(1-r)^2}{[\alpha + \beta(1-r)]^2} \tag{2.13}$$

<u>Result 1:</u> Better private information increases welfare.

<u>Result 2:</u> Better public information may not increase welfare for r sufficiently high.

Why? There is no problem with private information. But public information does two things. First, it increases the accuracy for forecasting θ which increases welfare. But then it also focuses people's beliefs away from θ through higher order expectations which coordinate people's actions by having them rely too much on the public signal relative to the private one. This tends to be the case when private information is sufficiently more accurate relative to the public one and when complementarities are sufficiently strong.

3 Sentiments

Idea: Use aggregate shocks on first-order beliefs ("sentiments") to produce business cycle fluctuations.

- essentially a theory about expectation formation
- maintains rational expectations
- delivers unique, but sunspot like equilibria
- can *increase* intrinsic volatility in the economy

Key Channel: Communication between economic actors about *idiosyncratic* fundamentals must be impeded so that common knowledge breaks down. Importantly, aggregate fundamentals still could be common knowledge.

3.1 Model Set-up

Basic idea is to build an island economy where islands are bilaterally matched to trade with each other.

Islands form expectations based on commonly known aggregate fundamentals, but only receive signals about trading partners *expectations*; i.e. higher order beliefs matter.

Production:

- $y_i = A_i n_i^{\theta} k^{1-\theta}$
- k is land and we can drop it by normalizing it to 1
- competitive firm on island maximizes profits as price taker
- A_i only source of heterogeneity

Household:

- consumption $u(c,c^*) = \left(\frac{c}{1-\eta}\right)^{1-\eta} \left(\frac{c^*}{\eta}\right)^{\eta}$
- leisure $v(n) = \frac{n^\epsilon}{\epsilon}$

Information:

- receive some signals
- make production decisions
- meet other island and exchange
- update information from exchange (rational expectation formation)

3.2 Equilibrium

Islands consume a fraction $(1-\eta)$ of production at home with η being exported.

Local prices (island i) and local terms of trade need to satisfy

$$p_i = \left(\frac{y_j}{y_i}\right)^{\eta} \quad (3.1)$$

$$p_i^* = \left(\frac{y_j}{y_i}\right)^{\eta-1} \quad (3.2)$$

$$R_i = p_i^{\frac{1}{\eta}} \quad (3.3)$$

and hence will depend on fundamentals of *both* islands.

Profit maximization before (!) exchange yields

$$n_i^{\epsilon-1} = \theta \frac{y_i}{n_i} E_{it}[p_i] \quad (3.4)$$

Hence: Individual forecasts for prices matter. For this I need to infer island j's productivity A_j since they spend $c_j = (1-\eta)y_j$ which influences my own price p_i.

Let island's type be denoted by ω. This type comprises its fundamentals (A_i) and the islands information set (E_i).

Result: A unique rational expectations equilibrium exists and satisfies the following fixed point problem

$$y(\omega) = \left(\theta^{\frac{\theta}{\epsilon}} A(\omega)\right)^{\frac{\epsilon}{\epsilon-\theta}} E[p_i|\omega] \quad (3.5)$$

$$E[p_i|\omega] = \int_{\Omega} p(\omega,\omega')\mathcal{P}(\omega'|\omega)d\omega' \quad (3.6)$$

$$p(\omega,\omega') = \left(\frac{y(\omega')}{y(\omega)}\right)^{\eta} \quad (3.7)$$

Why? Contraction mapping theorem.

3.3 A Game with Strategic Complementarities

This can be transformed into the following game among two islands picking their production y_i and y_j with the following payoff function

$$\log y_i = (1-\alpha)f_i + \alpha \mathbf{E}_i[\log y_j] \quad (3.8)$$

where $\mathbf{E}_i[X] = H^{-1}(E_i[H(X)])$ with $H(X) = \exp(\eta X)$.

This is exactly the formulation as before where (i) f_i depends on A_i and known parameters; (ii) $\alpha \in (0,1)$ is a function of known parameters; (iii) $\mathbf{E}_i$ is an adjusted expectations operator that depends on i's information set.

3.4 The Role of First-Order Belief Shocks (Sentiments)

3.4.1 Perfect Communication (Shared Beliefs)

Suppose islands have the *same belief* about each other's output level. Then, they must know each other's output.

Why? Island i knows A_i and its output level. So, by assumption island j must know it.

Rational expectations then imply that $\mathbf{E}_i = \log y_j$ and $\mathbf{E}_j = log_i$. Using this in the fixed point equation above, we get that

$$\log Y = \log \mathcal{B} = h(\bar{a}) \quad (3.9)$$

where $\bar{a} = \int \log A_i di$.

Aggregate Output is equal to average of islands' forecasts about the output of their trading partner and depends on fundamentals only.

3.4.2 Imperfect Communication

Information structure:

- $\log A_i \sim \mathcal{N}(0, \sigma_A^2)$
- sentiment shock $\xi \sim \mathcal{N}(0, \sigma_\xi^2)$
- two signals for island i

  1. signal about trading partner's output

  $$x_i^1 = \log A_j + u_{i1} \tag{3.10}$$

  where $u_{i1} \sim \mathcal{N}(0, \sigma_1^2)$

  2. signal about trading partner's belief about own productivity

  $$x_i^1 = x_j^1 + \xi + u_{i2} \tag{3.11}$$

  where $u_{i2} \sim \mathcal{N}(0, \sigma_2^2)$

Key: Sentiments are shocks on *first-order* beliefs and do not influence anyone's belief about aggregate or individual fundamentals.

Solution? Follow Morris and Shin (2002)!

Step 1: $\mathbf{E}_i[\log y_j] = E_i[\log y_j] + \frac{1}{2}\eta^2\sigma_y^2$

Step 2: Guess log-linear decision rule.

Step 3: Use to obtain island's i belief about $\log y_j$.

Step 4: Use the result in the fixed point equation and compare coefficients with the log-linear decision rule guessed in the first place.

Aggregate Output depends positively on ξ as does the average of islands' forecasts – or, the average sentiment $\mathcal{B}$ – about the output of their trading partner and depends on fundamentals only.

Why?

Shocks to ξ are simply "news" (a sunspot) that all *other* islands have received the news, too. This contains relevant information about the level of output on *all* islands and, hence, about demand.

If island i's output depends on A_i only, it would be optimal for island j to rely on A_j and x_j^1. But then island i would optimally (rational expectations(!)) rely on (i) x_i^1 as it contains information about A_j and (ii) on x_i^2 as it contains information about x_j^1. Sentiment shocks just move (exogenously) x_i^2 across all islands. It is then optimal to coordinate on this signal or, in other words, to coordinate on first-order beliefs.

3.4.3 Some Remarks

1) It is not even necessary to impose common knowledge as we have done above. All that is needed is to impose

$$\log E_i[p_i] = -\log E_j[p_j] \tag{3.12}$$

which means again that the two islands share a common belief about how their terms of trade – and, hence, their output – change.

2) One can introduce arbitrary belief shocks. However, these do not map 1-1 into observables so that there is indeterminacy. One can transform this belief structure into one that only has a unique (!) first-order, compound belief shock.

3) Sentiment shocks are very different from shocks to higher-order beliefs about fundamentals. Indeed, one can just propose any exogenous process for forming expectations and derive

an equilibrium given this process. Of course, this violates rational expectations.

4) What about news shocks? These are very different as they contain information about future fundamentals.

5) Suppose there are pure “noise” shocks, so that people have to make forecasts about fundamentals. In general, variations in output are limited by variations in fundamentals. But here they are not. Why? Because here sentiment shocks change forecasts about the specific actions of a single trading partner.

4 Literature

Veldkamp, Chapter 4 & 5 (2011)

Morris and Shin, AER (1998)

Morris and Shin, AER (2002)

Angeletos and La'O, Econometrica (2013)