Lecture Notes

Dynamic Contracts

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Fall Term 2014

1 Friction I – Limited Enforcement

1.1 Model

- Preferences:
$$E\left(\sum_{t=0}^{\infty}\beta^t u(c_t)\right) \tag{1.1}$$
- $\beta \in (0,1)$, u strictly increasing and strictly concave
- Endowment process: $\tilde{y}_t \in \{y_1, \ldots, y_S\}$, iid
- y_s with probability π_s

Perfect insurance gives lifetime utility equal to

$$V = \frac{1}{1-\beta}u(E(y_t)) \tag{1.2}$$

More generally, if there are many households and people can be forced to participate, any Pareto-optimal allocation is described by

$$\frac{1}{1-\beta}u(E(y_t) - \alpha_i) \tag{1.3}$$

and $\int \alpha_i di = 0$.

Problem:

If α_i is too high, person i has no incentive to participate in the insurance scheme (limited enforcement).

2 Model

- insurer (social planner) can borrow and lend at rate $r = \frac{1}{\beta} - 1$
- offers a contract to the household

- history: $h^t = (y_0, \ldots, y_t)$
- contract: $\{c_t\}_{t=0}^{\infty}$, where $c_t = f(h^t)$ for all t

Assumption:

Household can walk away from the insurance scheme at any time, but y_t is observable.

2.1 A Social Planning Problem

$$\max_{\{f(h^t)\}_{t=0}^{\infty}} E\left(\sum_{t=0}^{\infty} \beta^t (y_t - f(h^t))\right) \tag{2.1}$$

subject to

$$E\left(\sum_{t=0}^{\infty} \beta^t u(f(h^t))\right) \geq u_0 \tag{2.2}$$

$$u(f(h^t)) + \beta E_t\left(\sum_{j=1}^{\infty} \beta^{j-1} u(f(h^{t+j}|h^t))\right) \geq$$

$$u(y_t) + \beta E_t\left(\sum_{j=1}^{\infty} \beta^{j-1} u(y_{t+j})\right) \text{ for all } h^t, t \tag{2.3}$$

If $\tilde{y}_t$ is iid over time,

$$v_{aut} = E\left(\sum_{t=0}^{\infty} \beta^t u(y_t)\right) = E_t\left(\sum_{j=1}^{\infty} \beta^{j-1} u(y_{t+j})\right). \tag{2.4}$$

2.2 Solving – A First Try

Lagrangian:

$$E\left[\sum_{t=0}^{\infty} \beta^t \left\{(y_t - c_t) + \alpha_t \left(E_t\left[\sum_{j=0}^{\infty} \beta^j u(c_{t+j})\right] - u(y_t) - \beta v_{aut}\right)\right\}\right] + \phi E\left[\sum_{t=0}^{\infty} \beta^t u(c_t) - u_0\right] \tag{2.5}$$

Analysis:

—▷ Define $\mu_t = \mu_{t-1} + \alpha_t$, where $\mu_{-1} = 0$. This allows us to use the formula

$$\sum_{t=0}^{\infty} \beta^t \alpha_t \left(\sum_{j=0}^{\infty} \beta^j u(c_{t+j}) \right) = \sum_{t=0}^{\infty} \beta^t \mu_t u(c_t).$$

—▷ Then: we can solve a saddle-point problem given by

$$\max_{\{c_t\}} \min_{\{\mu_t\},\phi} E \left[\sum_{t=0}^{\infty} \beta^t \Big\{ (y_t - c_t) + (\mu_t + \phi) u(c_t) - (\mu_t - \mu_{t-1}) \left(u(y_t) - v_{aut} \right) \Big\} \right] - \phi u_0. \quad (2.6)$$

—▷ The solution is described by the FONC

$$u'(c_t) = \frac{1}{\mu_t + \phi}, \quad (2.7)$$

the condition that $\mu_t > \mu_{t-1}$ whenever the PC binds and $\phi > 0$. Note that μ_t is an endogenous stochastic process here that depends on the endowment shock.

—▷ Hence, whenever the PC is binding, consumption increases. Otherwise, it stays constant.

Remark:

The general idea behind this approach is that the Lagrange multiplier becomes a state variable keeping track of how binding the forward looking constraints were in the past. There are several issues with this approach, though. First, to be useful one looks for a recursive formulation. This formulation can be derived under weak conditions (see Marcet and Marimon (1998)). Second, solutions to the recursive formulation are only sufficient, but not necessary.

2.3 Solving – A Second Try

Problem:

—▷ contract keeps track of entire history: $c_t : H^t \rightarrow I\!R$

—▷ participation constraints are *forward-looking*

Idea:

Summarize entire history in a single state variable, which is a promised utility level, i.e.

$$u_{t+1} = E_t\left(\sum_{j=1}^{\infty}\beta^{j-1}u(c_{t+j})\right). \tag{2.8}$$

New Social Planning Problem:

$$V(u_0) = \max_{\{c_s,u_s\}_{s=1}^S}\sum_{s=1}^{S}\pi_s\left[(y_s - c_s) + \beta V(u_s)\right] \tag{2.9}$$

subject to

$$\sum_{s=1}^{S}\pi_s(u(c_s)+\beta u_s) \geq u_0 \tag{2.10}$$

$$u(c_s)+\beta u_s \geq u(y_s)+\beta v_{aut} \text{ for all } s\in S \tag{2.11}$$

$$c_s \in [\underline{c},\overline{c}] \text{ and } u_s\in[v_{aut},\overline{u}] \text{ for all } s\in S \tag{2.12}$$

Remark: Note that the participation constraints in conjunction with the restriction on u_s are key for being able to solve this problem. If the RHS of the participation constraint is not time invariant and if promised utility cannot be restricted to a compact interval, it is in general very hard to solve this recursive problem.

2.4 The Optimal Contract

FONC (off the boundary):

$$-\pi_s + \lambda\pi_s u'(c_s) + \mu_s u'(c_s) = 0 \tag{2.13}$$

$$\pi_s\beta V'(u_s) + \beta\lambda\pi_s + \mu_s\beta = 0 \tag{2.14}$$

Envelope condition:

$$V'(u_0) = -\lambda \tag{2.15}$$

Solution is characterized by three "equations":

$$u'(c_s) = -\frac{1}{V'(u_s)} \text{ for all } s \in S \tag{2.16}$$

$$V'(u_s) = V'(u_0) - \frac{\mu_s}{\pi_s} \tag{2.17}$$

$$\{s \in S | \mu_s > 0\} \cup \{s \in S | \mu_s = 0\} \tag{2.18}$$

—▷ the first equation equates the IMRS for the principal and the consumer

—▷ the second and third "equations" describe the wedge that the PC drives into the risk sharing problem

<u>Case 1: $\mu_s = 0$, i.e. non-binding PC</u>

—▷ $u_s = u_0$ and $u'(c_s) = -\frac{1}{V'(u_0)}$

—▷ constant consumption between periods for state s

—▷ why? $u_{t-1} = u_0$ implies that $u'(c_{t-1}) = -\frac{1}{V'(u_0)} = u'(c_{t,s})$ and u is strictly concave

<u>Case 2: $\mu_s > 0$, i.e. binding PC</u>

—▷ we have two equations in two unknowns

$$-u'(c_s)V'(u_s) = 1 \tag{2.19}$$

$$u(c_s) + \beta u_s = u(y_s) + \beta v_{aut} \tag{2.20}$$

—▷ since the PC is binding and V is strictly concave, we have $u_s > u_0 \geq v_{aut}$

—▷ the PC then implies that $c_s < y_s$

—▷ give up consumption today in exchange for higher future expected utility

—▷ history does not matter directly; only the current shock y_s matters ("amnesia")

—▷ history matters indirectly, however, as u_0 determines which PCs are binding

2.5 Dynamics

Question: Which constraints are binding?

Proposition 2.1. *There exists a cut-off state s' such that*

(i) $y_s < y_{s'} \Rightarrow \mu_s = 0$ *and*

(ii) $y_s \geq y_{s'} \Rightarrow \mu_s > 0$.

Proof. Suppose there exists $y_s > y_{s'}$ such that the PC is binding for s', but not for s. Then, $u_s = u_0$ and $u_{s'} > u_0$. Since the PC for state s is not binding, we have

$$u(c_s) + \beta u_s > u(c_{s'}) + \beta u_{s'}$$

which implies that $c_{s'} < c_s$ and $u'(c_{s'}) > u'(c_s)$.

Also, from the concavity of V, we have $-V'(u_{s'}) > -V'(u_s) = -V'(u_0)$. Hence,

$$-u'(c_{s'})V'(u_{s'}) > -u'(c_s)V'(u_s).$$

A contradiction. □

—▷ This implies that c_s and u_s are weakly increasing in y_s for any given u_0. Hence, the optimal contract raises *both* current consumption and future consumption in response to a contemporaneous endowment shock.

—▷ People with "good" shocks are constrained, as their outside option is high relative to their promised utility.

Proposition 2.2. *A (fictitious) cut-off value for when constraints are binding, $\bar{y}(u_0)$, is strictly increasing in u_0.*

Proof. The cut-off value is defined as

$$u(\bar{y}(u_0)) = u(c(u_0)) + \beta\left(u_0 - v_{aut}\right), \tag{2.21}$$

where $c(u_0)$ satisfies the FONC

$$-u'(c(u_0))V'(u_0) = 1. \tag{2.22}$$

Since V is concave, $c(u_0)$ is weakly increasing in u_0. □

—▷ Having realization $y_S = \max_{s \in S} y_s$ is an absorbing state with promised utility $\overline{u}$ forever and associated consumption level $\overline{c}$ such that

$$u(\overline{c}) + \beta\overline{u} = u(y_S) + \beta v_{aut} \tag{2.23}$$

and all other participation constraints being non-binding.

Remark: For all $u_0 > \overline{u}$, none of the PC is binding and a first-best is obtained.

3 Friction II – Limited Commitment

3.1 Model

- two agents
- uncertainty: $s \in S$, where s pins down the endowment distribution across agents
- $\{y_s^1, y_s^2\}_{s \in S}$, where $y_s^1 + y_s^2 = Y_s$
- π_s: probability of state s, iid
- "symmetric shocks": equal probability for each agent to obtain y_s

Assumption: Both agents can choose autarky forever at any time (two-sided limited commitment).

3.2 PCs and their Microfoundation

The PC are given by

$$u(c_t^i) + E_t \left[\sum_{\tau=1}^{\infty} \beta^\tau u(c_{t+\tau}^i) \right] \geq u(y_t^i) + \beta V_{aut} \tag{3.1}$$

for $i = 1, 2$ and for all t, s.

Again, an allocation is given by $\{c_t^1, c_t^2\}_{t=0}^{\infty}$, where $c_t^i : H^t \to [0, Y_s]$.

Question: What are the microfoundations for these constraints?

Idea:

- look at a "transfer game"
- people receive stochastic endowments each period and make history-dependent transfers to each other

- transfers are given by $\tau^i_{t,s} \in [0, y^i_{t,s}]$
- interpret allocation as resulting from people transfering part of their endowment
- allocation is feasible if $c^1_{t,s} + c^2_{t,s} \leq Y_s$
- allocation is incentive feasible if it is feasible and satisfies the PCs

Proposition 3.1. *An allocation is a subgame perfect equilibrium of the transfer game if and only if it is incentive feasible.*

Intuition:

Use "worst" equilibrium as a threat (Abreu, Econometrica (1988)).

1. Autarky is a subgame perfect equilibrium. Never transfer any endowment is a best response to never transfering any endowment.
2. Any other subgame perfect equilibrium must yield higher utility. One can always choose not to make any other transfers today and in the future. This gives the utility value of autarky.
3. Suppose now that an allocation satisfies (3.1) and is feasible. Then we can define transfers that lead to the allocation.

   Consider the following trigger strategy: make these transfers only if one has received transfers according to the allocation in the past; do not make any transfers again in the future, if one has not received these transfers in the past. These strategies constitute a subgame-perfect equilibrium.

4. Consider now transfers that correspond to some subgame perfect equilibrium. These transfers lead to an allocation (c^1, c^2). Then, at any date and state, an agent must be (weakly) better off with this allocation (or transfers) than with not making any

transfers today and never again. If he chooses not to make transfers today and never again, he achieves an alternative utility that is weakly better than

$$u(y^i_{t,s}) + \beta v_{aut}. \tag{3.2}$$

This implies that the utility derived from (c^1, c^2) is greater than this utility level.

3.3 Recursive Formulation of (Constrained) Efficient Allocations

$$V(u_0) = \max_{\{c_s,u_s\}_{s=1}^S} \sum_{s=1}^{S} [\pi_s u(Y_s - c_s) + \beta V(u_s)] \tag{3.3}$$

subject to

$$\sum_{s=1}^{S} \pi_s(u(c_s) + \beta u_s) \geq u_0 \tag{3.4}$$

$$u(c_s) + \beta u_s \geq u(y^1_s) + \beta v_{aut} \text{ for all } s \in S \tag{3.5}$$

$$u(Y_s - c_s) + \beta V(u_s) \geq u(Y_s - y^1_s) + \beta v_{aut} \text{ for all } s \in S \tag{3.6}$$

$$c_s \in [0, Y_s] \text{ and } u_s \in [v_{aut}, V_{max}] \text{ for all } s \in S \tag{3.7}$$

Remark: The value function V describes the (constrained) Pareto frontier for risk-sharing in this problem. The optimal contract $\{c_s, u_s\}_{s\in S}$ is "renegotiation-proof", in the sense that it describes an *efficient* subgame perfect equilibrium.

3.4 The Optimal Contract

We first show that only one PC can be binding in any state $s \in S$.

Proposition 3.2. *Let $S_i \subset S$ be the set of states where agent i's constraint is binding for an optimal contract. Then, $S_1 \cap S_2 = \emptyset$.*

Proof. Suppose there exists $s \in S_1 \cap S_2$. Since $u_s \geq v_{aut}$ and the constraint for agent 1 binds, we have $c_s \leq y^1_s$. Similarly, for agent 2, $Y_s - c_s \leq Y_s - y^1_s$. Hence, $c_s = y^1_s$, $u_s = v_{aut}$

and $V(u_s) = v_{aut}$. This implies that only the autarkic allocation is subgame perfect, since $V(v_{aut}) = v_{aut}$. A contradiction. $\square$

FONC:

$$\beta\pi_s V'(u_s) + \lambda\beta\pi_s + \beta\mu_s^1 + \beta\mu_s^2 V'(u_s) = 0 \tag{3.8}$$

$$-\pi_s u'(Y_s - c_s) + \lambda\pi_s u'(c_s) + \mu_s^1 u'(c_s) - \mu_s^2 u'(Y_s - c_s) = 0 \tag{3.9}$$

This implies that optimal contracts are given by

$$\frac{u'(Y_s - c_s)}{u'(c_s)} = -V'(u_s) \tag{3.10}$$

for all $s \in S$.

<u>Result:</u>

(i) Concavity of V implies that u_s is a non-decreasing function of c_s.

(ii) Hence, today's consumption and promised utility move together, i.e. good endowment shocks are smoothed out over time.

(iii) Binding participation constraints might prevent perfect smoothing across endowment realizations.

3.5 Dynamics

<u>Case 1: $s \notin S_1 \cup S_2$</u>

—▷ by the envelope theorem: $V'(u_0) = -\lambda$

—▷ $u_s = u_0$: allocation of consumption remains unchanged from last period

<u>Case 2: $s \in S_1$</u>

—▷ $\mu_s^1 > 0$ implies $V'(u_s) < V'(u_0)$

—▷ Concavity of V implies $u_s > u_0$: person 1's consumption increases from last period

Case 3: $s \in S_2$

—▷ $\mu_s^2 > 0$ implies $V'(u_s) > V'(u_0)$

—▷ Concavity of V implies $u_s < u_0$: person 1's consumption decreases from last period

3.6 History Dependence

The optimal contract exhibits

- positive correlation between current consumption and *current* income
- positive correlation between current consumption and *lagged* income

whenever the first-best is NOT a subgame-perfect equilibrium.

Why? Consider person 1:

—▷ if shocks cannot be smoothed over states, they are smoothed out over time

—▷ high endowment shocks cause PC to bind: u_s and c_s both increase

—▷ higher u_s means higher consumption in the future $c(u_s)$ from the FONC

Conclusion: Limited commitment endogenously introduces persistence into allocations.

3.7 Long-run Dynamics

Case 1: Some first-best allocation is incentive feasible.

Suppose there is no aggregate risk. Then, a first-best is incentive feasible if and only if[1]

$$\frac{1}{1-\beta}\sum_{s\in S}\pi_s u\left(\frac{Y}{2}\right) \geq u(y_{max}) + \beta v_{aut}. \tag{3.12}$$

[1] With aggregate shocks the condition would become

$$u\left(\frac{Y_s}{2}\right) + \frac{\beta}{1-\beta}\sum_{s\in S}\pi_s u\left(\frac{Y_s}{2}\right) \geq u(max\{y_s^1, y_s^2\}) + \beta v_{aut} \tag{3.11}$$

for all $s \in S$.

This is the case if

- β is close to 1
- u is very concave
- $Var[y]$ is large

Proposition 3.3. *For any initial condition $u_0 \in [v_{aut}, V(v_{aut})]$, the stochastic process $\{u_t\}_{t=0}^{\infty}$ converges w.p. 1 monotonically to the closest first-best allocation.*

Case 2: No first-best allocation is incentive feasible.

Proposition 3.4. *For any initial condition $u_0 \in [v_{aut}, V(v_{aut})]$, the stochastic process $\{u_t\}_{t=0}^{\infty}$ converges weakly to the same non-degenerate distribution of promised utility.*

3.8 Decentralizing Constrained Optimal Allocations

Kehoe and Levine (1993)

Idea:

- trading à la Arrow-Debreu
- however: trades must be enforced throughout time
- threat: permanent exclusion from trading in the future
- period 0 trades must be individual rational later on ("self-enforcing")

Household problem:

$$\max_{\{c_t(h^t)\geq 0\}} \sum_{t=0}^{\infty}\sum_{h^t} \beta^t \pi(h^t) u(c_t(h^t))$$

subject to

$$\sum_{t=0}^{\infty}\sum_{h^t} q_t^0(h^t)c_t(h^t) \leq \sum_{t=0}^{\infty}\sum_{h^t} q_t^0(h^t)y_t(h^t)$$

$$u(c_t(h^t)) + \beta E_t\left[\sum_{j=1}^{\infty}\beta^{j-1}u(c_{t+j})\right] \geq u(y_t(h^t)) + \beta E_t\left[\sum_{j=1}^{\infty}\beta^{j-1}u(y_{t+j})\right] \quad \text{for all } h^t, t$$

Result: If there is a single good, an allocation is constrained efficient if and only if it is a "constrained competitive equilibrium" with transfer payments.

—▷ decentralization of a "second-best"

—▷ key: constraint set of households is convex

—▷ critique: how is permanent exclusion enforced on markets?

Alvarez and Jermann (2002)

Idea:

- translate participation constraints into (sequential) borrowing constraints
- hence: restrictions on (negative) asset holdings
- sequentially complete markets with (endogenous) restrictions on size of trades

Recursive Household problem:

$$J_t(a, s^t) = \max_{c(s^t)\geq 0, \{a'(s_{t+1}, s^t)\}} u(c(s^t)) + \beta E\left[J_{t+1}(a', s_{t+1})\right]$$

subject to

$$c_t(s^t) + \sum_{s_{t+1}|s^t} q(s_{t+1}, s^t)a'(s_{t+1}, s^t) \leq y_t(s^t) + a$$

$$a'(s_{t+1}, s^t) \geq B_{t+1}(s_{t+1}, s^t) \quad \text{for all } s_{t+1}$$

Key refinement

$$J_{t+1}(B_{t+1}(s_{t+1}, s^t), s^{t+1}) = u(y_{t+1}(s^{t+1})) + \beta v_{aut} \tag{3.13}$$

Hence: Replace PCs with endogenous borrowing limits and restrict $a' < 0$ to a level that makes the household indifferent between paying back the debt and choosing his outside option.

Again, version of both Welfare Theorems hold.

Question: What are the effects on asset prices?

Proposition 3.5. *Let $\{c_i\}_{i\in I}$ be a constrained efficient allocation. If*

$$u(c^j(s_{t+1}, s^t)) + \beta E\left[\sum_{j=1}^{\infty} \beta^{j-1} u(c_{t+1+j})\right] > u(y^j(s_{t+1}, s^t)) + \beta v_{aut} \tag{3.14}$$

then

$$\frac{u'(c^j(s_{t+1}, s^t))}{u'(c^j(s^t))} = \max_{i\in I} \frac{u'(c^i(s_{t+1}, s^t))}{u'(c^i(s^t))}. \tag{3.15}$$

Unconstrained agents in tomorrow's state s_{t+1} have highest MRS and determine asset prices in equilibrium

$$q^*(s_{t+1}|s^t) = \beta\pi(s_{t+1}|s^t) \max_{i\in I} \frac{u'(c^i(s_{t+1}, s^t))}{u'(c^i(s^t))} \tag{3.16}$$

Intuition:

- unconstrained agents can alter their choice
- asset prices must be such that they have no incentive to do so
- a constrained agent j would like to borrow more in order to increase $c^j(s_t)$

$$q^*(s_{t+1}|s^t) > \beta\pi(s_{t+1}|s^t) \frac{u'(c^j(s_{t+1}, s^t))}{u'(c^j(s^t))} \tag{3.17}$$

- an agent is constrained when the endowment is high in state s^{t+1}

- he has then an incentive to default

Risk-free rate must be lower than in a complete markets environment. Why? Otherwise too much lending.

4 Friction III – Private Information

4.1 Model

We look now at an infinitely repeated principle agent problem with private information.

Principal

- risk-neutral, i.e. linear utility
- can borrow or invest at the constant gross interest rate $(1 + r) = 1/\beta$

Agent

- risk-averse with preferences represented by $u : (a, \infty) \longrightarrow I\!R$
- Assumptions: (i) $\sup u(c) < \infty$, (ii) $\inf u(c) = -\infty$, (iii) $\lim_{c \to a} u'(c) = \infty$, (iv) $-\frac{u''}{u'}$ is non-increasing
- Example: $u(c) = -\exp(-\alpha c)$

Private Information

- iid endowment process: y_s with probability π_s, where $s \in S$
- endowment realizations are private information
- agent makes reports to the principal about his state (direct mechanism)
- report in period t: $\sigma_t : S^t \longrightarrow S$
- reporting strategy: $\sigma = \{\sigma_t\}_{t=0}^{\infty}$
- truthful report: $\hat{\sigma}$, where $\hat{\sigma}_t(s^t) = s_t$ for all s^t and for all t (revelation principle)

4.2 Sequential Problem

A contract specifies consumption c_t for each possible history s^t. The optimal contract in sequential form is given by

$$V(u_0) = \max_{\{c_t(s^t)\geq 0\}} E\left[\sum_{t=0}^{\infty}\beta^t(y_t - c_t)\right] \tag{4.1}$$

subject to

$$E\left[\sum_{t=0}^{\infty}\beta^t u(c_t|\hat{\sigma}_t)\right] \geq u_0 \tag{4.2}$$

$$E\left[\sum_{\tau=t}^{\infty}\beta^\tau u(c_\tau|\hat{\sigma})\right] \geq E\left[\sum_{\tau=t}^{\infty}\beta^\tau u(c_\tau|\sigma)\right] \text{ for all } \sigma \text{ for all } t \tag{4.3}$$

The last constraint is the incentive compatibility constraint.

4.3 Recursive Problem

Question:

How can we formulate the incentive compatibility constraints in recursive fashion?

—▷ Green (1987): "temporary incentive compatibility constraints" (t.i.c)

—▷ no gains from one-period deviations from truth-telling:

$$u(c_t|\hat{\sigma}^t)+\beta E_t\left[\sum_{j=1}^{\infty}\beta^{j-1}u(c_{t+j+1}|\hat{\sigma}^{t+j+1})\right] \geq u(c_t|\sigma_t)+\beta E_t\left[\sum_{j=1}^{\infty}\beta^{j-1}u(c_{t+j+1}|(\hat{\sigma}_1,\ldots,\hat{\sigma}_{t-1},\sigma_t,\ldots,\hat{\sigma}_{t+j+1}))\right] \tag{4.4}$$

for all σ_t and all t.

Define $v_{max} = \frac{\sup u(c)}{1-\beta}$. The recursive problem of maximizing profits is then given by

$$V(u_0) = \sup_{\{b_s,u_s\}_{s=1}^S} \sum_{s=1}^S [\pi_s(-b_s) + \beta V(u_s)] \tag{4.5}$$

subject to

$$\sum_{s=1}^S \pi_s(u(y_s + b_s) + \beta u_s) = u_0 \tag{4.6}$$

$$u(y_s + b_s) + \beta u_s \geq u(y_s + b_k) + \beta u_k \text{ for all } s,k \in S \times S \tag{4.7}$$

$$b_s \in (a - y_s, \infty) \text{ and } u_s \in (-\infty, v_{max}] \text{ for all } s \in S \tag{4.8}$$

Remark:

The promise keeping constraint must hold exactly here. Intuitively, by granting a higher promised utility today than required, some t.i.c. in earlier periods might be violated.

Bounds on Value Functions:

—▷ a fixed payment of $\bar{b}(u_0)$ is incentive feasible, where $\bar{b}(u_0)$ solves

$$\frac{1}{1-\beta} \sum_{s=1}^S \pi_s u(y_s + \bar{b}) = u_0 \tag{4.9}$$

—▷ the first-best $\bar{c}(u_0)$ is NOT incentive feasible, but would deliver higher profits, since it is costly to provide incentives

—▷ bounds are given by

$$-\frac{\bar{b}(u_0)}{1-\beta} \leq V(u_0) < \frac{1}{1-\beta} \sum_{s=1}^S \pi_s(y_s - \bar{c}(u_0)) \tag{4.10}$$

—▷ one can show that V is concave and that these bounds imply that $\lim_{u_0 \to -\infty} V'(u_0) = 0$ and $\lim_{u_0 \to u_{\max}} V'(u_0) = -\infty$[2]

Assumption:

V is strictly concave and continuously differentiable everywhere.

[2]Due to the Inada condition it is cheap to increase u_0 for low levels of utility, but due to the upper bound it is expensive to increase u_0 for high levels of utility.

4.4 The Optimal Contract

One can show that only the "local downward" constraints bind, i.e. for the optimal contract

$$u(y_s + b_s) + \beta u_s = u(y_s + b_k) + \beta u_k \tag{4.11}$$

if and only if $k = s - 1$.

There is intertemporal insurance. Transfers are higher the lower the current income in exchange for lower future promised utility,

$$b_{s-1} \geq b_s \text{ and } u_{s-1} \leq u_s.$$

There is "co-insurance", i.e. both the principal and the agent benefit from higher income.

- Since local downward constraints (4.11) bind, we obtain

$$u(y_s + b_s) + \beta u_s = u(y_s + b_{s-1}) + \beta u_{s-1} > u(y_{s-1} + b_{s-1}) + \beta u_{s-1} \tag{4.12}$$

- It must be the case, that

$$-b_s + \beta V(u_s) \geq -b_{s-1} + \beta V(u_{s-1}). \tag{4.13}$$

  Why?

  Suppose not. Then set $b_s = b_{s-1}$ and $u_s = u_{s-1}$ (i.e. replace (b_s, u_s) by (b_{s-1}, u_{s-1})). This then increases profits. Since in any optimal contract only the local downward constraint (4.11) binds, this new contract satisfies incentive feasiblity and leaves total promised utility unchanged. This is a contradiction with the fact that the original allocation was optimal.

4.5 Martingale Property

<u>FONC:</u>

$$-\pi_s + \lambda \pi_s u'(y_s + b_s) + \mu_s u'(y_s + b_s) - \mu_{s+1} u'(y_{s+1} + b_s) = 0 \tag{4.14}$$

$$\pi_s \beta V'(u_s) + \lambda \pi_s \beta + \mu_s \beta - \mu_{s+1} \beta = 0 \tag{4.15}$$

for all $s = 1, \ldots, S$, where $\mu_1 = 0$ and, by convention, $\mu_{S+1} = 0$.

Proposition 4.1. *$V'(u_0)$ is a martingale.*

Proof. From the envelope condition, we obtain that $V'(u_0) = -\lambda$. Sum over the second FONC with respect to s to obtain

$$\sum_{s=1}^{S} \pi_s \beta V'(u_s) + \lambda\beta = 0. \tag{4.16}$$

Hence, $E\left[V'(u_{t+1})\right] = V'(u_t)$. □

Hence: It is optimal to equate today's marginal costs of delivering u_0 to the expected costs of delivering $(u_1, \ldots, u_S)$ tomorrow.

4.6 Dynamics

The key result is that the agent gets "impoverished" with probability one. That is promised utility drifts downwards over time. The optimal contract itself smooths consumption, but at the same time increases variability of promised future utility.

Proposition 4.2. *$u_t \to -\infty \quad a.s.$*

Proof. Step 1:

$u_N > u_0 > u_1$, i.e. continuation values spread out over time. This is a direct consequence of the FONC, the fact that $b_{s-1} \geq b_s$, $u_{s-1} \leq u_s$ and concavity of V.[3]

Step 2:

V' is a non-positive martingale. By the Martingale Convergence Theorem, $V' \to \xi$ almost surely. As V' is strictly concave and $\lim_{u_0 \to -\infty} V'(u_0) = 0$, it suffices to show that $\xi = 0$ almost surely.

[3]In fact, after a long enough sequence of positive (negative) shocks, the agent becomes a creditor (debtor).

Step 3:

Suppose to the contrary, that $\lim_{t\to\infty} V'(u_t) = \xi < 0$, implying that u_0 converges almost surely to some finite level $\tilde{u}$. It suffices to derive a contradiction for a set of paths $(s_1, s_2, \dots)$ that occur w.p. 1. Note that the set of paths where $s = N$ occurs only finitely many times has probability 0. Hence, we rule out those paths.

Take any sequence of promised utility over time. Take a subsequence $\{u_t\}$ where $s = N$ for all t. There exists then a convergent subsequence $\tilde{u}_t \to \tilde{u}$. Denote by g the law of motion of the stochastic process describing the evolution of u_t, or

$$u_{t+1} = g(u_t, s) \tag{4.17}$$

where g is continuous by the Theorem of the Maximum. Hence, $\lim_{t\to\infty} g(\tilde{u}_t, s = N) = g(\tilde{u}, N)$. Since V is continuously differentiable, we have

$$V'(\tilde{u}) = \lim_{t\to\infty} V'(\tilde{u}_{t+1}) = \lim_{t\to\infty} V'(g(\tilde{u}_t, s = N)) = V'(g(\tilde{u}, N)). \tag{4.18}$$

Hence, by strict concavity of V, we have $\tilde{u}_0 = \tilde{u}_N$ which contradicts Step 1. □

4.7 Some final remarks

—▷ Atkeson and Lucas (1992) and (1995) characterize optimal allocations for a situation with many agents and a closed economy. Their characterization is based on a decentralization of optimal allocations. Hence, it is based on an appropriate version of the Second Welfare Theorem. The main difficulty in these papers arises from the fact that the distribution of promised utility is a state variable.

—▷ Other papers allow for the accumulation of assets. Examples are Cole and Kocherlakota (2001) which introduce hidden storage and Ligon, et al. (2000) that introduce storage into the two-sided limited commitment model.

—▷ Comparison between the long-run dynamics of the three contracting models:

- One-sided limited commitment: Flat identical consumption profile
- Two-sided limited commitment: Stable distribution of consumption inequality (for $S > 2$ and no first-best incentive feasible)
- Hidden information: Almost all people become impoverished