Economics 817 **Winter 2017**

Advanced Macroeconomic Theory II **Thorsten Koeppl**

Assignment 1

(Due: Monday, January 23)

<u>Note:</u> `Hand in a print-out of your program and send the program also by e-mail to the TA. Make sure to add comments on each step in the code to express what the code is doing.`

1. Consider the metric space $(I\!R^2, ||\cdot||)$, where $||\cdot||$ is the sup-norm defined by $\max_{i=1,2} |x_i - y_i|$ for $\mathbf{x} = (x_1, x_2)' \in I\!R^2$ and $\mathbf{y} = (y_1, y_2)' \in I\!R^2$.

   Define the operator $T : I\!R^2 \longrightarrow I\!R^2$ by

   $$T\mathbf{x} = \alpha + \beta \mathbf{P}\mathbf{x},$$

   where $\alpha = (\alpha_1, \alpha_2)'$, $\beta \in (0,1)$ and

   $$\mathbf{P} = \begin{pmatrix} P_1 & 1 - P_1 \\ 1 - P_2 & P_2 \end{pmatrix}$$

   where $P_1, P_2 \in [0,1]$.

   (a) Prove that T has a unique fixed point.

   (b) Set $\alpha = (2,1)$, $\beta = 0.9$ and $(P_1, P_2) = (0.8, 0.5)$. Solve numerically for the fixed point of the operator T defined by $T\mathbf{x} = \mathbf{x}$. To do so, compute a sequence $\{\mathbf{x}_n\}_{n=0}^{N}$, where the initial value is given by $\mathbf{x}_0 = (0,0)$ and $\mathbf{x}_n$ is defined by

   $$\mathbf{x}_{n+1} = \alpha + \beta \mathbf{P}\mathbf{x}_n.$$

   Stop computing this sequence when $\max_{i=1,2} |x_{n,i} - x_{n+1,i}| < 0.00001$. Compute directly the fixed point from the formula $\mathbf{x} = (I - \beta \mathbf{P})^{-1}\alpha$. Print out both the first ten elements of the sequence$\{\mathbf{x}_n\}_{n=0}^{N}$ and the computed fixed point $\mathbf{x}$.

(c) Can you be sure that your computer program finds the fixed point approximately? Does this fix point depend on the initial starting value of $\mathbf{x}$? Explain your answer.

(d) The maxtrix $\mathbf{P}$ is called a *Markov Chain*. In the example of part (b) what is the *stationary distribution* and *ergodic set* associated with $\mathbf{P}$? [Hint: SLwP, Ch. 11 or S+L, Ch. 2]

(e) Find values for P_i such that the Markov Chain $\mathbf{P}$ does **not** have a unique stationary distribution.

2. Consider the problem of an infinitely lived household that has linear utility, $u(c) = c$, discounts the future according to the discount factor β and maximizes his lifetime utility

$$\sum_{t=0}^{\infty} \beta^t c_t.$$

Suppose further that the household can borrow or lend at a (net) interest rate equal to $r = \frac{1}{\beta} - 1$. Denote the level of assets the household owns at time t by b_{t-1} taking as given the initial level of assets b_{-1}.

Consider two cases. First, the household is not restricted in the amount he can borrow. Second, the household cannot borrow at all, or $b_t \geq 0$ for all t.

(a) For both cases, formulate the SP and FE that describes the households decision problem.

(b) For both cases, do Theorem 4.2 - 4.5 in SLwP hold? Explain your answer.

3. Consider now the following problem. Let x be the state variable, with the return function $F(x, y) = x - \beta y$. There is only one feasible plan given by

$$\Gamma(x) = \begin{cases} -\frac{x}{\beta} \text{ if } x \geq 0 \\ \frac{x}{\beta} \text{ if } x < 0. \end{cases}$$

Note that the constraint set is a singleton, i.e. for each x there is a unique continuation state y.

(a) Formulate the SP and FE for this problem.

(b) Do Theorems 4.2 - 4.5 apply for this problem?

4. Consider a social planner that chooses a sequence of capital $\{k_{t+1}\}_{t=0}^{\infty}$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t \ln c_t$$

subject to

$$k_{t+1} + c_t = A k_t^{\alpha}$$

$$c_t \geq 0, k_{t+1} \geq 0$$

where $A > 0$, $\alpha, \beta \in (0, 1)$.

(a) Set up the problem of the planner recursively.

(b) Guess that the value function is given by

$$V(k) = a_0 + a_1 \ln k$$

and verify that this is correct for appropriately chosen constants a_0 and a_1.

(c) Find the policy function that describes the optimal capital accumulation.