Economics 815 **Winter 2020**

Macroeconomic Theory **Thorsten Koeppl**

Answer Key for Assignment 6

Answer to Question 1:

1. The household solves the problem

$$\max_{(C_t,N_t,B_t)} E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{\chi_t C_t^{1-\sigma}}{1-\sigma} + \frac{(1-N_t)^{1-\eta}}{1-\eta} \right)$$

subject to

$$P_t C_t + Q_t B_t \leq W_t N_t + B_{t-1} + T_t.$$

Note that P_t and C_t are aggregates as defined in the Lecture Notes. Furthermore, the price of a one-period nominal (discount) bond with zero coupon is given by Q_t. Note that χ_t is a preference shock that changes aggregate demand.

To show once again clearly how to derive the Euler equation, I assume that uncertainty can be described by probabilities over states in each period. Denote $\pi(s^t)$ as the probability of the history of states $(s_0, s_1, \ldots, s_t)$.

The FOCs are given by

$$\begin{aligned}
\pi(s^t)\beta^t C(s^t)^{-\sigma}\chi(s^t) &= \lambda(s^t)P(s^t) \\
\pi(s^t)\beta^t (1-N(s^t))^{-\eta} &= \lambda(s^t)W(s^t) \\
-\lambda(s^t)Q(s^t) + \sum_{s^{t+1}} \lambda(s^{t+1}|s^t) &= 0,
\end{aligned}$$

where the last one is with respect to $B(s^t)$ and the summation is over successor states s^{t+1} of history s^t.

We obtain that

$$Q(s^t)\pi(s^t)\beta^t \frac{C(s^t)^{-\sigma}}{P(s^t)}\chi(s^t) = \sum_{s^{t+1}} \pi(s^{t+1})\beta^{t+1} \frac{C(s^{t+1})^{-\sigma}}{P(s^{t+1})}\chi(s^{t+1})$$

or

$$1 = E_t \left[\beta \left(\frac{\chi_{t+1}}{\chi_t} \right) \left(\frac{C_t}{C_{t+1}} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \frac{1}{Q_t} \right].$$

2. It is useful to define the inflation rate $\Pi_{t+1} = \frac{P_{t+1}}{P_t}$ and the nominal interest rate $Q_t = \frac{1}{1+i_t}$. Thus, we have that

$$1 = E_t \left[\beta \frac{\chi_{t+1}}{\chi_t} \left(\frac{C_t}{C_{t+1}} \right)^{-\sigma} \frac{1}{\Pi_{t+1}} (1 + i_t) \right].$$

Denote for any variable $\hat{x}_t = \log X_t - \log X_{SS} = x_t - x_{SS}$. Log-linearizing both sides of the equation – use the rules from the lecture – we obtain

$$-\sigma \hat{c}_t + \hat{\chi}_t = E_t \left[-\sigma \hat{c}_{t+1} + \hat{\chi}_{t+1} - \hat{\pi}_{t+1} + \widehat{1 + i_t} \right].$$

In steady state, we have that $C_t = C_{t+1} = C_{SS}$ and $\chi_t = \chi_{SS}$ so that the Euler equation is given by

$$\frac{1}{\beta} = \Pi_{SS}(1 + \bar{\iota})$$

or defining $\rho = -\log \beta$,

$$\rho = \pi_{SS} + \log(1 + \bar{\iota}) \simeq \pi_{SS} + \bar{\iota}.$$

Rewriting the log-linearized Euler equation we get

$$-\sigma c_t + \log(\chi_t) = E_t[-\sigma c_{t+1} + \log(\chi_{t+1}) - (\pi_{t+1} - \pi_{SS}) + (\log(1 + i_t) - \log(1 + \bar{\iota})].$$

Using the SS relationship and noting that $\log(1 + i_t) \simeq i_t$ we obtain

$$c_t - E_t[c_{t+1}] = \frac{1}{\sigma} \left(\log(\chi_t) - E_t[\log(\chi_{t+1})] \right) - \frac{1}{\sigma} \left(i_t - E_t[\pi_{t+1}] - \rho \right).$$

Now log-linearize the market clearing condition around the steady state $Y_{SS} = C_{SS} + G_{SS}$. We obtain

$$Y_{SS} \log \left(\frac{Y_t}{Y_{SS}} \right) = C_{SS} \log \left(\frac{C_t}{C_{SS}} \right) + G_{SS} \log \left(\frac{G_t}{G_{SS}} \right).$$

Changing notation and dividing by Y_{SS}, we get

$$\hat{y}_t = \frac{C_{SS}}{Y_{SS}} \hat{c}_t + \frac{G_{SS}}{Y_{SS}} \hat{g}_t = s_c \hat{c}_t + s_g \hat{g}_t$$

where s_c and s_g are shares of private and public consumption in steady state.

Substituting into the log-linearized Euler equation from above, we obtain

$$y_t - E_t[y_{t+1}] = \frac{s_c}{\sigma}\left(\log(\chi_t) - E_t[\log(\chi_{t+1})]\right) - \frac{s_c}{\sigma}\left(i_t - E_t[\pi_{t+1}] - \rho\right) - s_g E_t[g_{t+1} - g_t].$$

Define now $r_t^n = \rho + \frac{\sigma}{s_c} E_t[y_{t+1}^n - y_t^n]$. Then, we obtain the IS equation in terms of the output gap and the natural rate of interest as

$$x_t - E_t[x_{t+1}] = -\frac{s_c}{\sigma}\left(i_t - E_t[\pi_{t+1}] + r_t^n\right) + \frac{s_c}{\sigma}\left(\log(\chi_t) - E_t[\log(\chi_{t+1})]\right) - s_g E_t[g_{t+1} - g_t].$$

Hence, changes in the output gap are a function of the Euler equation and changes in the shocks to preferences/private demand and government spending.

Remark: One can incorporate gov't spending into the framework along the following lines. Gov't demand for each individual good is given by

$$G_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\epsilon} G_t$$

with the derivation identical to the one given in the lecture notes for private demand. Total lump-sum taxes T in nominal terms are taking the place of private nominal expenditures Z_t in the derivations. Since taxes are lump-sum, none of the analysis changes so that total demand for good i is given by $C(i) + G(i)$.

3. Since there are no technology shocks, we have that $y_{t+1}^n - y_t^n = 0$, so that $r_t^n = \rho$. This implies that the Euler equation becomes

$$x_t - E_t[x_{t+1}] = \frac{s_c}{\sigma}\left(\log(\chi_t) - E_t[\log(\chi_{t+1})]\right) - \frac{s_c}{\sigma} E_t[\pi_{t+1}] - s_g E_t[g_{t+1} - g_t].$$

We guess and verify a solution (see below for more on this). Set

$$g_t = -\frac{s_c}{s_g \sigma}\log(\chi_t).$$

for all t. Since χ_t is the only shock, we have that $x_t = 0$ and $\pi_t = 0$ for all t satisfies both the NKPC and the IS equation. Hence, we have an equilibrium. Government expenditures exactly offset fluctuations in private demand. If aggregate private demand increases (falls), government expenditures fall (increase).

Remark: Even though we have found an equilibrium with no output gap and zero inflation, this equilibrium will not be unique. From the NKPC, we have that a zero output gap for all t yields

$$\pi_t = \beta E_t[\pi_{t+1}] = \beta^2 E_t[E_{t+1}[\pi_{t+2}]] = \beta^2 E_t[\pi_{t+2}] = \dots.$$

In principle, this admits many solutions, so that we have indeterminancy. We simply picked the solution that has $\pi_t = 0$ for all t. A similar problem would occur for the IS equation, where any process with $E_t[x_{t+1}] = 0$ would lead to indeterminancy with respect to the output gap, but not with inflation which would be pinned down by the exogenous variations in the output gap according to

$$\pi_t = \kappa x_t + \kappa \sum_{k=1}^{\infty} \beta^k E_t[x_{t+k}] = \kappa x_t.$$

To avoid such a problem of indeterminacy, we would need to formulate again how fiscal expenditures react to variations in x_t and π_t.

4. Calibration is not a precise exercise. One could for example use consumption data to fit an AR(1) process to obtain the parameters (ρ_χ and the standard deviation of the shock. A similar procedure could be used for gov't expenditures. Alternatively, one could choose these four parameters that match (imperfectly) second-order moments of a simulated economy to such moments in the data. More sophisticated methods rely on Bayesian estimation of parameters which `DYNARE` can also handle quite easily.

   Note also that you need to pick a number for s_g. This number can easily be obtained as a percentage of gov't expenditure of total output or GDP.

5. We have that deviations in output y_t are identical to deviations in the output gap x_t, since $r_t^n = \rho$ and y_t^n is constant due to the absence of technology shocks. The output shows the rest of the variables for a 1% increase in χ. The responses are as expected. We have an increase in consumption and, hence, a positive output gap. Inflation increases with a positive response in nominal interest rates (see Figure 1 below).

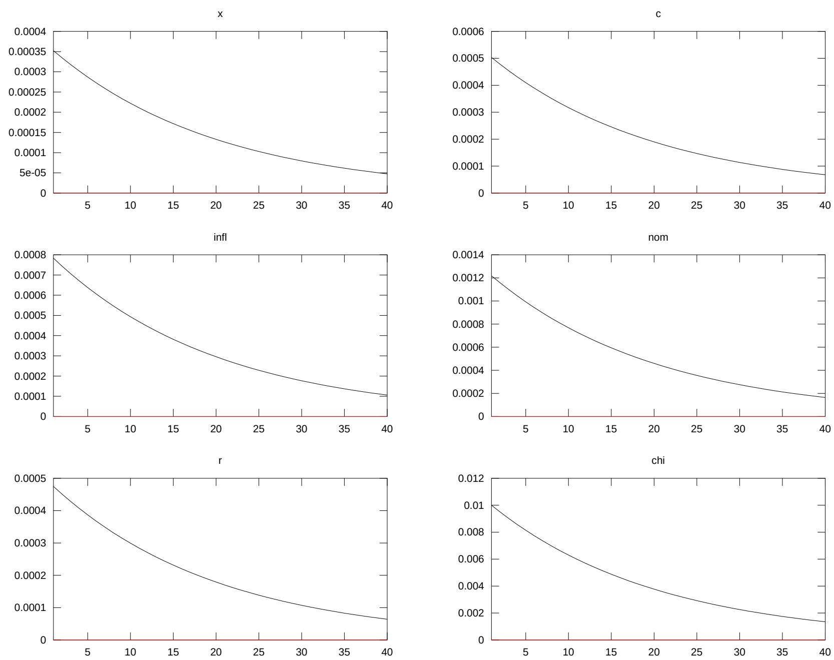

Figure 1: IRFs for Taste Shock

6. The responses are the same as in part (e) except for consumption. Higher gov't expenditures crowd out private consumption. The strength of this effect depends on your calibration of s_g, which I set to 30% of output (see Figure 2 below).

7. Increasing ϕ_π makes the policy response to demand shocks more aggressive. As a result, both inflation and the output gap are more stabilized. This shows that with demand shocks there is no trade-off between inflation and output stabilization. Reacting very strongly to inflation achieves the lowest variability in both variables – and, hence, the highest welfare as pointed out in class (see Figure 3 below). This is often referred to as the divine coincidence of monetary policy in New Keynesian economics.

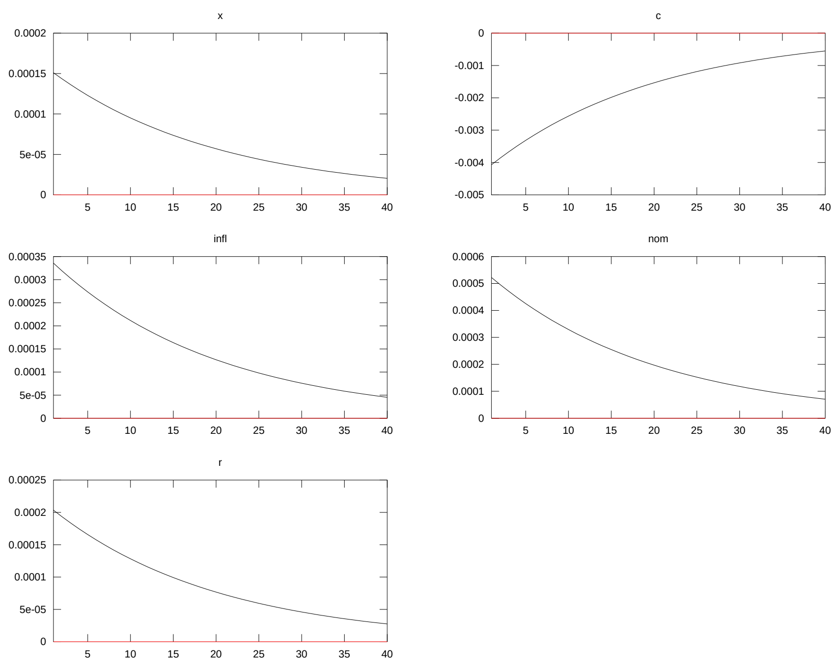

Figure 2: IRFs for Gov't Spending Shock – $\phi_y = 0.125$

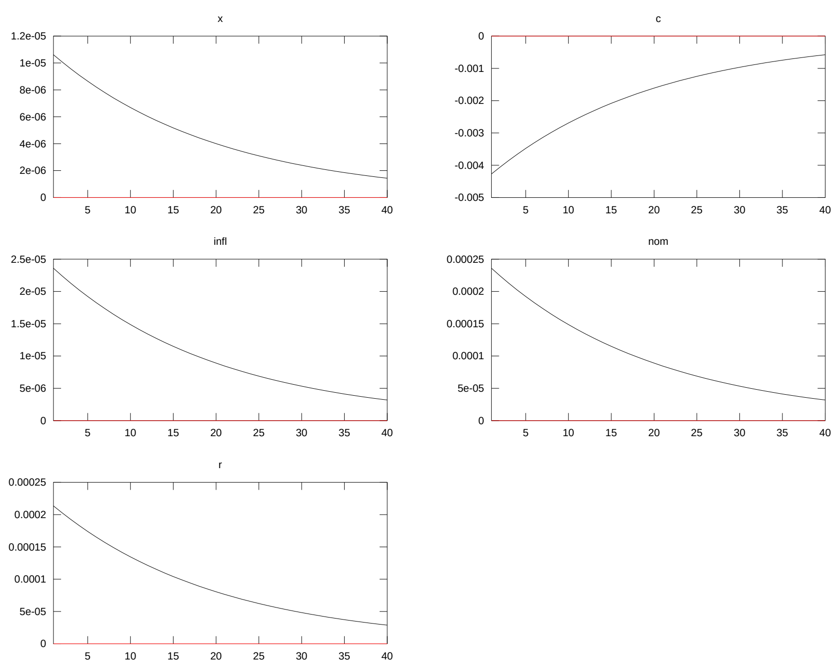

Figure 3: IRFs for Gov't Spending Shock – $\phi_y = 0$

Answer to Question 2:

1. The optimization problem is given by

$$\min_{\phi_\pi} E_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\alpha x_t^2 + \pi_t^2 \right) \right]$$

subject to

$$x_t = f_x(\epsilon_t, u_t)$$

$$\pi_t = f_\pi(\epsilon_t, u_t)$$

where the functions f_x and f_π are given by the matrix equation in the question.

Note that $E_0[\epsilon_t] = E_0[u_t] = 0$ so that the objective function can be rewritten as

$$L = \sum_{t=0}^{\infty} \beta^t (\alpha E_0[x_t^2] + E_0[\pi_t^2]) = \frac{1}{1-\beta} \left(\alpha Var[x_t] + Var[\pi_t] \right).$$

Hence, we need to determine the variance terms for the matrix equation we have found in part (b). Since the shocks are uncorrelated and the means for x_t and π_t are normalized to 0, we have

$$\begin{aligned} Var(x_t) &= \left(\frac{\sigma}{\sigma + \kappa \phi_\pi} \right)^2 \sigma_\epsilon^2 + \left(\frac{\phi_\pi}{\sigma + \kappa \phi_\pi} \right)^2 \sigma_u^2 \\ Var(\pi_t) &= \left(\frac{\sigma \kappa}{\sigma + \kappa \phi_\pi} \right)^2 \sigma_\epsilon^2 + \left(\frac{\sigma}{\sigma + \kappa \phi_\pi} \right)^2 \sigma_u^2. \end{aligned}$$

Neglecting constant terms, the problem can thus be rewritten as[1]

$$\min_{\phi_\pi} \left(\frac{1}{\sigma + \kappa \phi_\pi} \right)^2 \left[\alpha (\sigma^2 + \sigma^2 \kappa^2) \sigma_\epsilon^2 + (\phi_\pi^2 + \sigma^2) \sigma_u^2 \right].$$

Remark: The matrix equation can be obtained by plugging the policy rule for i_t into the the IS equation and solving forward. See the hand-out and use $z_t = (x_t, \pi_t)$ for the vector of jump or control variables and $\eta_t = (\epsilon_t, u_t)$ for the vector of shocks.

[1] Note that σ is a preference parameter (intertemporal elasticity of substitution), whereas σ_ϵ and σ_u refer to the standard deviation of the two shocks.

2. The first-order condition yields[2]

$$\phi_\pi^* = \sigma\kappa \left[\frac{1}{\alpha} + \left(\frac{\alpha + \kappa^2}{\alpha} \right) \left(\frac{\sigma_\epsilon}{\sigma_u} \right)^2 \right].$$

3. The parameter α is a welfare weight on output gap ("unemployment") relative to inflation variability. The lower this weight, the more aggressive is the response to inflation differing from 0. Inflation targeting can be seen as a low weight α and, thus, the prescription for such a regime is to respond aggressively to inflation.

   Note that only the relative variance of the two shocks matters for given α. If demand shocks (ϵ) increase, the prescription is to react more strongly. However, for supply shocks (u_t), exactly the opposite is the case: one should not respond strongly in situations where supply shocks are relevant (i.e. their variance is high).

   Finally, κ is inversely related to θ, the degree of price stickiness. If θ is high – say close to 1 – firms cannot change their prices. Hence, inflation pressures are low. In such a case, κ will be low which implies that the reaction coefficient ϕ_π should also be set low.

[2]One can easily verify that at this value of ϕ_π the second-order condition is strictly positive.

Answer to Question 3:

Remark:

In my calibration, I use a log-log specification on preferences. This implies that the natural rate of output moves 1-1 with a technology shock.

Negative Permanent Technology Shock:

I do not plot the impulse response function to a permanent, negative productivity shock. The reason is that with a permanent shock, both the natural level of output and actual output jump by exactly 1% down on impact and remain there. There is no transition.

It is easy to check that all other variables do not move. The real interest rate, the nominal interest rate and the natural rate of interest are all constant and equal to ρ. Inflation is always zero. This also implies that the interest rate rule does not matter at all, provided it ensures determinancy.

Intuitively, distortions in the model only arise since firms that can adjust their prices at different times face a different level of productivity at that time. With a permanent shock, no one has an incentive to adjust prices given that the shock is permanent. Hence, relative prices across firms are fixed and so is relative output between different goods.

Suppose now, that people learn in period 0 that there will be a negative technology shock in period 5. The IRFs are shown below for the benchmark Taylor rule.

- In period 5, the shock materializes. Output and the natural level of output drop by 1% immediately and we are back in the permanent shock scenario.

- At $t = 0$, news arrives and output reacts immediately. Note that the natural rate of interest initially remains constant. The shock has not materialized yet. By definition, it needs to jump down in the period before the shock hits.

- Output declines over time which can be explained by consumption smoothing from the Euler equation. This implies that the real interest rate needs to be below the SS level as there is negative consumption growth.

- The nominal interest rate and inflation react according the Phillips Curve and the Taylor Rule. Demand falls and, hence, inflation is below 0. This causes a reaction in

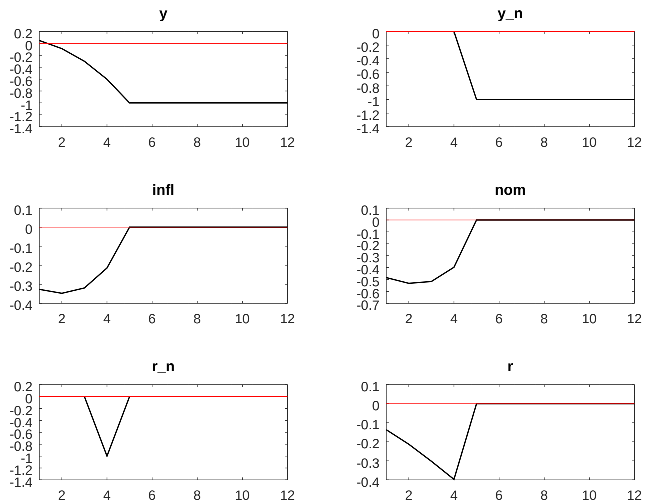

Figure 4: Response to News about Negative Technology Shock

the nominal interest rate that is larger than the drop in inflation. Consequently, real interest rates drop which – in equilibrium – stabilizes demand.

- Increasing the coefficient on inflation leads to a response where output stays close to the initial SS for a while. This is efficient here. The same happens if one increases the coefficient on the output gap in the Taylor Rule. Check it out!

Remark: Something is puzzling here. Some firms decrease prices in the first 4 periods. Hence, when the permanent technology shock materializes, we have a non-degenerate price distribution (indeed, we have a total of 5 different prices!).
Firms can still change their prices. For zero inflation some firms must lower their prices while other firms must increase their prices. This can only be consistent with the model. Firms that have lowered their prices increase them and firms that have not lowered their prices in the first four periods decrease them.
Also note that if we introduce some history dependence such as an AR(1) process on the shock or inertia in interest rates according to a Taylor Rule of the form

$$i_t = \rho + \rho_i i_{t-1} + \phi_\pi \pi_t + \phi_x x_t$$

we will get overshooting at Period 5 for inflation. In a sense, firms that have decreased prices try to raise them again.

<u>Positive Cost Push Shock:</u>

The cost push shock shifts the Phillips Curve. Here, the shift is permanent. Again, I am not showing the responses to an immediate shock. They are subsumed in the picture below.

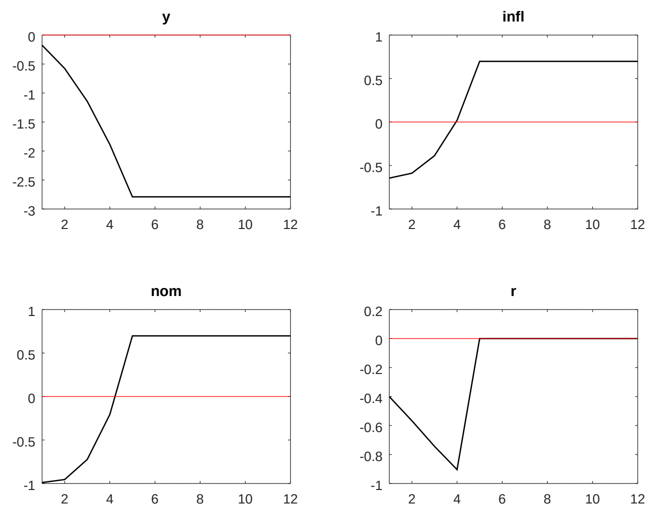

Figure 5: Response to News about Cost Push Shock

- In period 5, the economy immediately transitions to the new SS. First, notice that the natural rate of output is not affected by the cost push shock at all. From the Phillips Curve equation, either inflation or the output gap must be positive (or both).

- The Taylor Rule will drive how the economy reacts to the shock. In the graph below, we will have both higher inflation and a negative output gap. The nominal interest rate in the long-run adjusts to keep the real interest rate at ρ.

- In the transition, inflation falls as output declines causing a response from the central bank to lower the nominal interest rate. The real rate declines, stabilizing somewhat the fall in demand by discouraging savings.

- Increasing the response coefficient to inflation dampens the demand response and, hence, the output response in the short-run. It also lowers inflation in the long-run at the expense of a large output response.

  Increasing the response coefficient to output is very interesting. First, it causes a bigger immediate reaction, but mutes the dynamics. Second, it increases the inflation response, but lowers the output response.

  This gives us a trade-off how to react to cost push shocks. The relative size of the reaction coefficients matters.

Coronavirus Scenario:

It is natural to think of the coronovirus as a negative productivity shock due to global closures and lockdowns, resulting in a halt in global production of a large proportion of goods and services.
This negative shock has in turn resulted in consumers hoarding certain products such as masks, disinfecting and hygiene-related products, foods with long shelf lives etc. Governments are also have a difficult time over the procurement of ventilators and PPE. We can model this as a positive demand shock.
On the flip side, consumer spending in restaurants, leisure, and some retail trade sectors has fallen as a result of precautionary measures to curb the spread of the virus (i.e. negative demand shock in other sectors). It is not clear which effect is stronger. Hence, we assume a smaller initial, negative demand shock that becomes deeper over time.
Finally, we recognize that there may be inflationary pressures building over time due to reorganizing the economy. This is captured by a delayed, permanent cost push shock. To the contrary, we assume that there the productivity and demand shocks are not permanent and go away within 12 quarters.

The table below summarizes the transition path for the shocks.

Quarter	1	2	3	4	5	6	7	8	9	10	11
a	-1	-1	-0.75	-0.5	-0.5	-0.5	-0.5	-0.25	0	0	0
e	-0.25	-0.5	-1	-1	-1	-1	-0.5	-0.5	-0.25	0	0
u	0	0	0	0	0	0	0.25	0.5	0.5	0.5	0.5

The IRFs are given in the graph below.

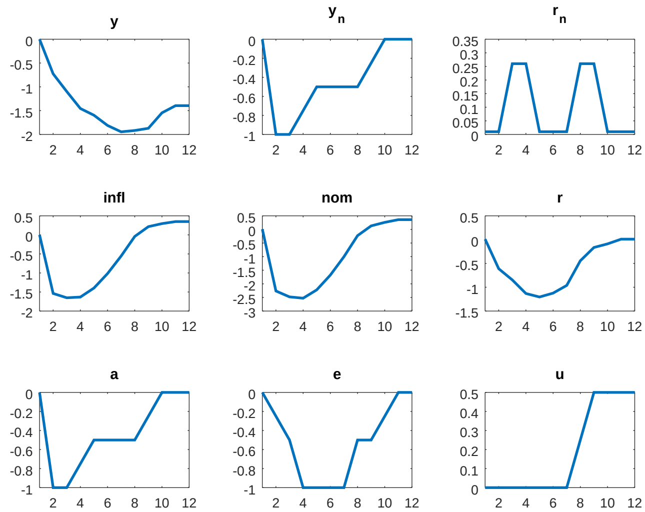

Figure 6: CoVid-19 simulation