Economics 815 **Winter 2020**

Macroeconomic Theory **Thorsten Koeppl**

Answer Key for Assignment 3

Answer to Question 1:

(a) From our lecture notes, we know that income is Pareto distributed. The number of people with income above some threshold y is thus given by the survival function of the Pareto Distribution

$$1 - F(y) = y^{-\frac{\delta}{\mu}}$$

where we have normalized the minimum income to $y_0 = 1$ the total population to 1. Importantly, we need to assume that $\delta \geq \mu$.

We first derive what share of total income is made up of incomes above some threshold $\tilde{y}$. We then can infer how many people have incomes above this threshold from the Pareto distribution. This allows us to pin down the share of income that goes to people above a certain income level.

The total income going to incomes above $\tilde{y}$ is given by

$$\begin{aligned}
\int_{\tilde{y}}^{\infty} y dF(y) &= \int_{\tilde{y}}^{\infty} y d\left(1 - y^{-\frac{\delta}{\mu}}\right) \\
&= \frac{\delta}{\mu} \int_{\tilde{y}}^{\infty} y^{-\frac{\delta}{\mu}} dy \\
&= \left(\frac{\delta/\mu}{1 - \delta/\mu}\right) y^{1-\frac{\delta}{\mu}} \Big|_{\tilde{y}}^{\infty} \\
&= \left(\frac{\delta}{\delta - \mu}\right) \tilde{y}^{1-\frac{\delta}{\mu}}
\end{aligned}$$

We can derive the total income in the population by simply setting $\tilde{y} = y_0 = 1$. Hence, the share of total income in the population going to incomes above $\tilde{y}$ is given by

$$s = \tilde{y}^{1-\frac{\delta}{\mu}}.$$

We know that a fraction $n = \tilde{y}^{-\frac{\delta}{\mu}}$ have incomes above the threshold. Hence, for income level $\tilde{y}$, $n^{-\frac{\mu}{\delta}}$ people have income above it. Thus, the share of total income going to the percentage n of top earners is given by

$$s = (\tilde{y}(n))^{-\frac{\delta}{\mu}} = n^{1-\frac{\mu}{\delta}}.$$

(b) For the plot of a Lorenz curve, see the lecture notes.

The Gini coefficient is defined by twice the area between the 45^o degree line and the Lorenz curve. To calculate it, we integrate the function s on the interval $[0, 1]$ and substract the integral of n over the same interval from it. We get

$$\begin{aligned} G &= 2\int_0^1 \left(n^{1-\frac{\mu}{\delta}} - n\right) dn \\ &= 2\left(\frac{\delta/\mu}{2\delta/\mu - 1} - \frac{1}{2}\right) \\ &= \frac{1}{2\frac{\delta}{\mu} - 1} \\ &= \frac{\mu}{2\delta - \mu} \end{aligned}$$

(c) I am using data from the conference board which have the Gini coefficient rise from 0.29 to 0.31 and 0.32 in terms of after-tax income inequality. This is important, as a redistributive tax system mitigates some of the earnings inequality.

Our model points to two potential channels for a higher Gini coefficient (aka more inequality). Either top talent has become relatively more scarce (δ decreases) or the premium that talent earns has increased (μ increases) over this period or both. This is of course exactly what the model is supposed to deliver by construction.

Answer to Question 2:

(a) The social planner's problem is given by

$$\max_{C_t, K_{t+1}, H_{t+1}} = \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\gamma}}{1-\gamma} \right)$$

subject to

$$C_t + X_{kt} + X_{ht} \leq A K_t^{\alpha} H_t^{1-\alpha}, \quad \forall\, t$$

$$K_{t+1} = X_{kt} + (1-\delta) K_t, \quad \forall\, t$$

$$H_{t+1} = X_{ht} + (1-\delta) H_t, \quad \forall\, t$$

K_0 and H_0 given.

We can represent this problem with the Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\gamma}}{1-\gamma} \right) + \lambda \left(A K_t^{\alpha} H_t^{1-\alpha} - C_t + (1-\delta) K_t - K_{t+1} + (1-\delta) H_t - H_{t+1} \right).$$

The Euler equations are

$$1 = \beta \left(\frac{C_t}{C_{t+1}} \right)^{\gamma} (F_{k,t+1} + 1 - \delta) \text{ for all } t$$

$$1 = \beta \left(\frac{C_t}{C_{t+1}} \right)^{\gamma} (F_{h,t+1} + 1 - \delta) \text{ for all } t.$$

By equating the FOCs we also have

$$F_{kt} + 1 - \delta = F_{ht} + 1 - \delta, \quad \forall\, t$$

$$\frac{\alpha}{1-\alpha} H_t = K_t, \quad \forall\, t.$$

This simplifies the Euler equations to

$$1 = \beta \left(\frac{C_t}{C_{t+1}} \right)^{\gamma} \left(A \alpha^{\alpha} (1-\alpha)^{1-\alpha} + 1 - \delta \right) \text{ for all } t.$$

(b) The growth rate in the balanced growth path where $g = g_y = g_k = g_h = g_c$ can be solved using any of the Euler equations using $1 + g_c = \frac{C_{t+1}}{C_t}$. Hence,

$$
\begin{aligned}
1 &= \beta \left(\frac{C_t}{C_{t+1}} \right)^{\gamma} \left(A\alpha^{\alpha} (1-\alpha)^{1-\alpha} + 1 - \delta \right) \\
\left(\frac{C_{t+1}}{C_t} \right)^{\gamma} &= \beta \left(A\alpha^{\alpha} (1-\alpha)^{1-\alpha} + 1 - \delta \right) \\
(1 + g_c)^{\gamma} &= \beta \left(A\alpha^{\alpha} (1-\alpha)^{1-\alpha} + 1 - \delta \right) \\
g = g_c &= \beta \left(A\alpha^{\alpha} (1-\alpha)^{1-\alpha} + 1 - \delta \right)^{\frac{1}{\gamma}} - 1.
\end{aligned}
$$

(c) For the balanced growth path, the production function takes the form $Y_t = AK_t$. This can be seen as follows. Using $\frac{1-\alpha}{\alpha} K_t = H_t$ and the production function

$$
\begin{aligned}
Y_t &= AK_t^{\alpha} H_t^{1-\alpha} \\
&= AK_t^{\alpha} \left(\frac{1-\alpha}{\alpha} K_t \right)^{1-\alpha} \\
&= A \left(\frac{1-\alpha}{\alpha} \right)^{1-\alpha} K_t \\
&= \mathcal{A} K_t,
\end{aligned}
$$

where $\mathcal{A} = A(\frac{1-\alpha}{\alpha})^{1-\alpha}$. Note, the economy could equally be represented by $Y_t = \mathcal{B} H_t$, where $\mathcal{B} = A(\frac{\alpha}{1-\alpha})^{\alpha}$.

Remark: Hence, this economy reduces to the so-called AK-model along the balanced growth path.

Remark: Note that the transition to the balanced growth path do not necessarily need to satisfy this relationship between physical and human capital. Indeed, one would expect there to be corner solutions where investment is 0 into one of the factors. The reason is that it is always efficient to invest into the factor that has the highest marginal product. I would suspect that the economy would be converging very fast to a balanced growth path, putting all investment first into the factor that is too low relative to the BGP.

(d) The savings rate of physical capital in the balanced growth path is given by

$$\begin{aligned} s_{kt} &= \frac{K_{t+1} - (1-\delta)K_t}{Y_t}, \\ &= \frac{K_{t+1} - (1-\delta)K_t}{\mathcal{A}K_t} \\ &= \frac{1}{\mathcal{A}}\left(\frac{K_{t+1}}{K_t} - 1 + \delta\right) \\ &= \frac{1}{\mathcal{A}}\left(g_k + \delta\right). \end{aligned}$$

Symmetrically, the savings rate for human capital is given by $s_{ht} = \frac{1}{\mathcal{B}}\left(g_h + \delta\right)$. Since in the balanced growth path the two growth rates have to be identical, the total saving rate in the economy is thus given by $s_t = s_{kt} + s_{ht} = \left(\frac{1}{\mathcal{A}} + \frac{1}{\mathcal{B}}\right)(g + \delta)$.

Answer to Question 3:

(a) The social planner's problem is given by

$$\max_{c_1, c_2, k} (1-\beta)\log c_1 + \beta \log c_2$$

subject to

$$c_1 + c_2 + k = f(k)$$

Note that this looks like a static problem and, indeed, here it is for the social planner. The planner has to take into account that he has to rebuild the steady state capital stock and that he has to allocate consumption between the young generation (c_1) and the old generation (c_2).

However, it is purely by assumption (!) that the planner optimizes the utility of a representative generation.

(b) Taking first order conditions, we obtain

$$\frac{c_2}{c_1} = \frac{\beta}{(1-\beta)}$$

which pins down the allocation of resources within a period between the two generations.

The optimal capital stock is given by

$$f'(k) = Ak^{\alpha-1} = 1$$

or

$$k = (\alpha A)^{\frac{1}{1-\alpha}}$$

Combining the Euler equation and aggregate resource constraint we have

$$\begin{aligned} c_1 &= Ak^{\alpha} - k - c_2 \\ &= k^{\alpha}(A - k^{1-\alpha}) - \frac{\beta}{1-\beta}c_1 \\ &= (\alpha A)^{\frac{\alpha}{1-\alpha}}(1-\alpha)A - -\frac{\beta}{1-\beta}c_1 \end{aligned}$$

Hence, due to the log utility, consumption is split as a fraction of β and $(1-\beta)$ between the two generations, or

$$\begin{aligned} c_1 &= (1-\beta)(\alpha A)^{\frac{\alpha}{1-\alpha}}(1-\alpha)A = (1-\beta)\phi(k^*) \\ c_2 &= \beta(\alpha A)^{\frac{\alpha}{1-\alpha}}(1-\alpha)A = \beta\phi(k^*) \end{aligned}$$

where $\phi(k^*)$ is output net of investment at the optimal capital stock which is often referred to the *golden rule level of capital.*

(c) The household's problem is given by

$$\max_{c_{t,t}, c_{t,t+1}, k_t} (1-\beta)\log(c_{t,t}) + \beta \log(c_{t,t+1})$$

$$\text{subject to}$$

$$c_{t,t} = w_t - k_t$$

$$c_{t,t+1} = r_{t+1}k_t$$

where w_t is the wage paid and hence labour income from supplying one unit of labour and r_{t+1} is the interest rate earned next period from renting out capital acquired through savings k_t when young. The first index refers to the generation and the second one to time.

In what follows, we surpress these indices as we will be in steady state. Consumption c_1 and c_2, however, have to be interpreted to be for the *same generation*.

After setting up the Lagrangian and taking first order conditions we get

$$\frac{c_2}{c_1} = \frac{\beta}{1-\beta} r_t$$

which is the intertemporal Euler equation. Note that – unless $r_t = 1$ – this equation is NOT the same as the rule for the social planner how to split consumption across two generations in any period.

(d) Rearranging for c_2 in the Euler equation and using a life-time budget constraint we get

$$c_1 = (1-\beta)w_t$$
$$c_2 = \beta w_t r_t$$
$$k_t = \beta w_t$$

Savings are *independent* of the interest rate which is a consequence of log utility where income and substitution effects just cancel out.

The wage rate is given by the marginal product of labour from the firm's maximization problem:

$$w_t = (1-\alpha)A k_t^{\alpha}$$

Hence, in the steady state, we have

$$k = \beta w = (\beta A(1-\alpha))^{\frac{1}{1-\alpha}}$$

The interest rate is then given by the marginal product of capital from the firm's maximization problem so that

$$r = \alpha A k^{\alpha-1} = \frac{\alpha}{(1-\alpha)\beta}$$

so that in general $r \neq 1$.

Finally, consumption is then given by

$$c_1 = (1-\beta)(1-\alpha)A\left(\beta A(1-\alpha)\right)^{\frac{\alpha}{1-\alpha}}$$
$$c_2 = \left(\frac{\beta\alpha}{(1-\alpha)\beta}\right)(1-\alpha)A\left(\beta A(1-\alpha)\right)^{\frac{\alpha}{1-\alpha}}$$

(e) We can simply compare the equilbrium interest rate with the MPK of the planning solution. Hence, we need to have that

$$r = \frac{\alpha}{(1-\alpha)\beta} = 1$$

If $\alpha/(1-\alpha) = \beta$, the two solutions are identical.

If $\alpha/(1-\alpha) > \beta$, there is underaccumulation of capital as the equilibrium interest rate is large than the MPK of the planning solution.

If $\alpha/(1-\alpha) < \beta$, in equilibrium the economy is saving too much so that there is overaccumulation. This is the situation where transfers (or public debt) helps reduce savings and can achieve a more efficient dynamic allocation of resources.

Answer to Question 4:

(a) The household's maximization problem is given by

$$\max_{c_t;k_{t+1};n_t} = E_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\gamma}}{1-\gamma} + \theta \frac{(1-n_t)^{1-\eta}}{1-\eta} \right) \right]$$

subject to

$$c_t + x_t \leq w_t n_t + r_t k_t, \quad \forall\, t \text{ and } z_t$$

$$k_{t+1} = x_t + (1-\delta)k_t, \quad \forall\, t \text{ and } z_t$$

$$k_0 \text{ and } z_0 \text{ given.}$$

The firm takes wages and interest rates as given and solves the static problem

$$\max_{n_t,k_t} z_t k_t^{\alpha} k_t^{1-\alpha} - w_t n_t - r_t k_t, \quad \forall\, t \text{ and } z_t.$$

The households maximization problem can be represented with the Lagrangian

$$\mathcal{L} = E_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\frac{c_1^{1-\gamma}}{1-\gamma} + \theta \frac{(1-n_1)^{1-\eta}}{1-\eta} \right) + \lambda \left(w_t n_t + r_t k_t - c_t + (1-\delta)k_t - k_{t+1} \right) \right].$$

The FOCs for the household's problem are

$$\frac{(1-n_t)^\eta}{c_t^\gamma} = \frac{\theta}{w_t(z_t)}, \qquad \forall\, t \text{ and } z_t$$

$$1 = \beta E_t\left[\left(\frac{c_t}{c_{t+1}}\right)^\gamma (r_{t+1} + (1-\delta))z_t\right], \qquad \forall\, t \text{ and } z_t$$

$$c_t + x_t = w_t(z_t)n_t + r_t(z_t)k_t + (1-\delta)k_t, \qquad \forall\, t \text{ and } z_t.$$

The FOC for the firm's problem is given by

$$r_t(z_t) = \alpha y_t(z_t)/k_t, \qquad \forall\, t \text{ and } z_t$$

$$w_t(z_t) = (1-\alpha)y_t(z_t)/n_t, \qquad \forall\, t \text{ and } z_t.$$

Finally, this leads to the steady state equations

$$\frac{(1-\bar{n})^\eta}{\bar{c}^\gamma} = \frac{\theta}{f_n}$$

$$1 = \beta(f_k + 1 - \delta)$$

$$\bar{c} = \bar{z}F(\bar{k}, \bar{n}) - \delta\bar{k}.$$

(b) Alternative calibrations will lead to varying dynamics, which is expected as long as the calibration is well motivated. Use $\gamma = \eta = 1$ (or $\in [1,5]$). Lecture 7 provides the rational for appropriate quarterly ranges for $\beta \in [0.96, 0.99]$, $\delta \in [0.0125, 0.025]$, and $\alpha \in [0.2, 0.3]$. For θ, the weight on the utility function, between consumption and leisure, one can use the strategy outlined in Hansen (JME, 1985). Use the three steady state conditions and calibrations for the parameters β, δ and α and calibrate to *levels* as pointed out in the lecture slides. Levels are normalized by setting the productivity level to $\bar{z} = 1$. Begin by finding a steady state value for hours worked, n, from the data expressed as a percentage of available time, which has been normalized to 1, usually $\bar{n} \in [0.2, 0.3]$. This allows us to determine the steady state level of capital k from

$$1 = \beta(F_k + 1 - \delta)$$

$$1 = \beta(\alpha\bar{k}^{\alpha-1}\bar{n}^{1-\alpha} + 1 - \delta)$$

$$\bar{k} = \left(\frac{1/\beta + \delta - 1}{\alpha\bar{n}^{1-\alpha}}\right)^{\frac{1}{\alpha-1}},$$

and determine the consumption level c from

$$\begin{aligned}c + k &= F(k, n) + (1 - \delta)k \\ \bar{c} &= \bar{k}^{\alpha}\bar{n}^{1-\alpha} - \delta\bar{k}.\end{aligned}$$

Taking as given the values for γ and η, we can now use the first-order condition for the leisure choice to solve for the value of θ:

$$\begin{aligned}\frac{c^{-\gamma}}{\theta(1-n)^{-\eta}} &= \frac{1}{F_n} \\ \frac{\bar{c}^{-\gamma}}{\theta(1-\bar{n})^{-\eta}} &= \frac{1}{(1-\alpha)\bar{k}^{\alpha}\bar{n}^{-\alpha}} \\ \theta &= \frac{\bar{c}^{-\gamma}(1-\alpha)\bar{k}^{\alpha}\bar{n}^{-\alpha}}{(1-\bar{n})^{-\eta}}.\end{aligned}$$

(c) Steady state values for $(k; c; y)$ and θ can be calibrated as described above. The AR(1) TFP process is easiest calculated using de-trended labor productivity (GDP over hours worked), or using the log-difference (percent change) of an institutional measure of labor or total factor productivity. As discussed in lecture 7, values of $\sigma \in [0.0015, 0.006]$ are appropriate along with a persistent AR(1) process, $\rho \in [0.95, 0.98]$.

The major difference between the data and the model is the absence of the government and net exports in the national accounts. Due to this, the aggregate ratios for c/y and k/y will generally be higher than the corresponding shares in GDP for the Canadian economy. Over the last 40 years for the Canadian economy, consumption is within 56% to 60% of GDP and investment is within 15% to 30% of GDP. As with the investment share, the capital share in steady state will also be amplified relative to the data. In the data, capital to GDP for the Canadian economy is usually 200% to 400% but is regarded as an imprecise measure. Note in the steady state of the model, the relation between the capital stock and investment comes from the perpetual inventory equation:

$$\begin{aligned}k_{t+1} &= (1-\delta)k_t + x_t \\ \bar{k} &= (1-\delta)\bar{k} + \bar{x} \\ \bar{k} &= \frac{\bar{x}}{\delta}.\end{aligned}$$