**Economics 815** **Winter 2020**

**Macroeconomic Theory** **Thorsten Koeppl**

## Answer Key for Assignment 2

**Answer to Question 1:**

1. The household maximizes utility taking the interest rate and profits as given

$$\max_{c_1,c_2,s} u(c_1,c_2) = \frac{c_1^{1-\gamma}}{1-\gamma} + \beta \frac{c_2^{1-\gamma}}{1-\gamma}$$

subject to

$$c_1 + s \leq y$$

$$c_2 \leq rs + \Pi$$

where $s$ denotes savings by the household and $\Pi$ is the profit from the firm.

Remark: Interpreting savings as investment, we could write here $r + (1-\delta)$ for the return after depreciation. Then, $1 + r$ is the gross return on investment. Using $\delta = 1$, we get the above formulation.

2. The firm's problem is to maximize profits taking the interest rate as given

$$f(k) = k^\alpha - rk.$$

3. A competitive equilibrium for this economy is an interest rate $r$ and an allocation $(c_1, c_2, s, k)$ such that

   (a) households maximize utility taking the interest rate and profits as given

   (b) firms maximize profits taking the interest rate as given

   (c) markets clear

$$\begin{aligned} c_1 &= y - k \\ c_2 &= k^\alpha \\ s &= k. \end{aligned}$$

4. From the firm's decision problem, we obtain

$$f'(k) = \alpha k^{\alpha-1} = r.$$

¿From the consumer's problem we obtain

$$\begin{aligned} c_1^{-\gamma} &= \lambda_1 \\ \beta c_2^{-\gamma} &= \lambda_2 \\ -\lambda_1 + r\lambda_2 &= 0. \end{aligned}$$

This yields the intertemporal Euler equation

$$\left(\frac{c_2}{c_1}\right)^{\gamma} = \beta r.$$

Now we can use the market clearing conditions with $y = 1$ to obtain

$$\left(\frac{k^{\alpha}}{1-k}\right)^{\gamma} = \beta\alpha k^{\alpha-1}.$$

5. Using the parameters of the model we can solve the following non-linear equation for $k$

$$k^{\alpha\gamma} - (1-k)^{\gamma}\beta\alpha k^{\alpha-1} = 0.$$

The solution is given by

$$\begin{aligned} k^* &= 0.23963 \\ c_1^* &= 0.76037 \\ c_2^* &= 0.65142 \\ r^* &= 0.81553. \end{aligned}$$

6. The graph below shows how the equilibrium values vary with the elasticity of intertemporal substitution $\gamma$.

7. The interpretation is straightforward. The coefficient $\gamma$ expresses both risk aversion and the inverse of the elasticity of intertemporal substitution. A higher $\gamma$ implies a lower elasticity, which means that households have a stronger preference to smooth

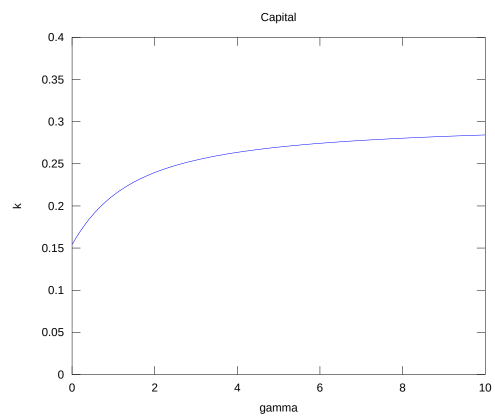



Figure 1: Capital as a function of CRRA coefficient

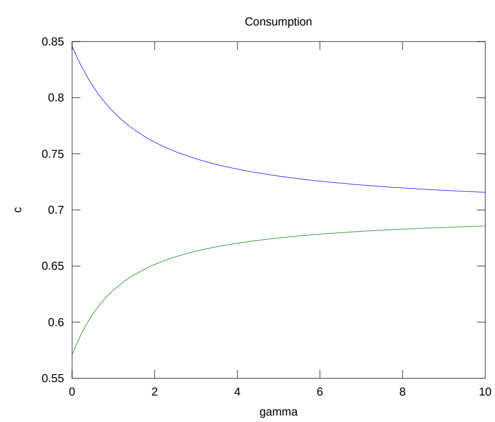



Figure 2: Consumption as a function of CRRA coefficient

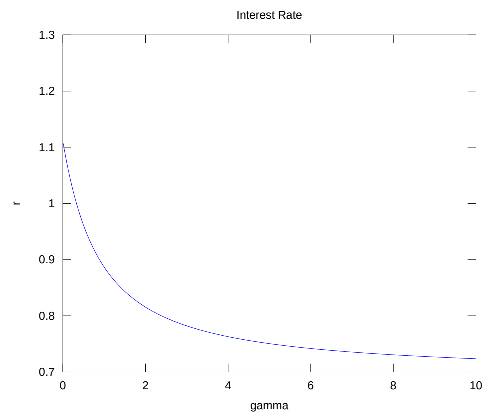



Figure 3: Interest rate as a function of CRRA coefficient

consumption over time. The only way for consumers to smooth their consumption is to invest into capital that is productive tomorrow. Consequently, capital is larger which lowers its marginal product tomorrow and, hence, the interest rate. Hence, the larger $\gamma$, the lower the interest rates will be.

Note that there is a general equilibrium effect. Lower interest rates – for any given $\gamma$ – will discourage consumption growth, and, hence savings. This will dampen the effect of a change in $\gamma$ on consumption growth. The effect would not be present, if interest rates were constant due to a constant return in capital or savings. Then, a larger $\gamma$ would decrease consumption growth one-for-one.

## Answer to Question 2:

1. Denote today's probability distribution across states as $p_t$ and tomorrow's probability distribution across states as $p_{t+1}$. Since today's state is $\underline{y}$, we have $p_t = (1, 0)$.

   We have

$$p_{t+1} = p_t \Pi.$$

Hence, the probability of $\underline{y}$ is given by $1 \cdot 0.8 + 0 \cdot \mu$ and the one of $\bar{y}$ by $1 \cdot 0.2 + 0 \cdot (1-\mu)$, so that $p_{t+1} = (0.8, 0.2)$.

2. Let $p^* = (p, q)$ denote the long-run stationary distribution.

   The long-run stationary distribution solves

$$p^* = p^* \Pi.$$

   This yields the following equations

$$\begin{aligned} p &= 0.8p + \mu q \\ q &= 0.2p + (1-\mu)q \\ 1 &= p + q. \end{aligned}$$

   Note that the first and second equations are linearly dependent, so that we can use only one of the equations. We can use the third equation with $p \in (0,1)$ to obtain $q = 1 - p$. Therefore, $p = \frac{\mu}{0.2+\mu}$ and $q = 1 - \frac{\mu}{0.2+\mu}$ for $\mu \in (0,1]$. For example, when $\mu = 1$, the long-run distribution is given by $p = \frac{5}{6}$ and $q = \frac{1}{6}$.

   A special case is given by $\mu = 0$, where we obtain a stationary distribution given by $p^* = (0,1)$. This means that $\bar{y}$ is an *absorbing state*.

**Answer to Question 3:**

One could start off with a brute force approach and maximize the following Lagrangian

$$\begin{aligned} E_0 & \left[ \frac{c_0^{1-\gamma}}{1-\gamma} + \beta \frac{c_1^{1-\gamma}}{1-\gamma} + \beta^2 \frac{c_2^{1-\gamma}}{1-\gamma} \right] \\ & + \lambda_0 \left[ y_0 - c_0 + q_H^0 a_H - q_L^0 a_L - q_{HH}^0 a_{HH} - q_{LH}^0 a_{LH} - q_{HL}^0 a_{HL} - q_{LL}^0 a_{LL} \right] \\ & + \lambda_H \left[ \bar{y} - c_H + a_H^1 + q_{HH}^1 (a_{HH} - \tilde{a}_{HH}) + q_{HL}^1 (a_{HL} - \tilde{a}_{HL}) \right] \\ & + \lambda_L \left[ \underline{y} - c_L + + a_L^1 + q_{LH}^1 (a_{LH} - \tilde{a}_{LH}) + q_{LL}^1 (a_{LL} - \tilde{a}_{LL}) \right] \\ & + \lambda_{HH} \left[ \bar{y} - c_{HH} + \tilde{a}_{HH} \right] \\ & + \lambda_{LH} \left[ \bar{y} - c_{LH} + \tilde{a}_{LH} \right] \\ & + \lambda_{HL} \left[ \underline{y} - c_{HL} + \tilde{a}_{HL} \right] \\ & + \lambda_{LL} \left[ \underline{y} - c_{LL} + \tilde{a}_{LL} \right] \end{aligned}$$

Some comments. Note that the state in period 0 is known, so that we have simply used the notation $y_0$. Next, AD securities that pay out in period 2 are retraded at a price $q^1$ in period 1. Hence, these assets pay no dividend in period 1, but have a price at its payoff. The constraints express the *net trades* in these securities. Finally, in period 2, all AD securities have zero prices as the world ends.

Alternatively, one could look at a sequential problem and work backwards. For example, suppose the state is $H$ in period 1. Then, we have

$$\max_{c,a} E_1 \left[ \frac{c_H^{1-\gamma}}{1-\gamma} + \beta \frac{c_2^{1-\gamma}}{1-\gamma} \right]$$

subject to

$$c_H = \bar{y} + q^1_{HH} a_{HH} + q^1_{HL} a_{HL}$$

$$c_{HH} = \bar{y} + a_{HH}$$

$$c_{HL} = \underline{y} + a_{HL}$$

and a similar problem for the state $L$. Once again, we can solve a period 0 problem taking the period 1 as given.

It is important to realize that some of the period 2 AD securities have a price of 0 in period 1 conditional on that state in that period. This has been taken into account in the formulation of the problem already. For example, $q^1_{LH} = 0$ if the state in period 1 is $H$. But $q^1_{LH} > 0$ if the state in period 1 is $L$.

1. The FOCs yield

$$c_0^{-\gamma} = \lambda_0$$

$$\frac{\beta}{2} c_i^{-\gamma} = \lambda_i \quad \text{for } i = H, L$$

$$\frac{\beta^2}{4} c_{ij}^{-\gamma} = \lambda_{ij} \quad \text{for } i, j = H, L$$

$$\lambda_0 q_i^0 = \lambda_i \quad \text{for } i = H, L$$

$$\lambda_0 q_{ij}^0 = \lambda_i q_{ij}^1 \quad \text{for } i, j = H, L$$

$$\lambda_i q_{ij}^1 = \lambda_{ij} \quad \text{for } i, j = H, L$$

We can now get rid of the Lagrange multipliers to obtain an expression for every asset price. For example, period 0 asset prices for AD securities in period 1 are given by

$$q_i^0 = \frac{\beta}{2}\left(\frac{c_0}{c_i}\right)^\gamma$$

Using the market clearing condition, we have that

$$\begin{aligned}
q_H^0 &= \frac{\beta}{2}\left(\frac{y_0}{\bar{y}}\right)^\gamma \\
q_L^0 &= \frac{\beta}{2}\left(\frac{y_0}{\underline{y}}\right)^\gamma
\end{aligned}$$

Similarly, we have for the AD securities in period 2

$$\begin{aligned}
q_{HH}^0 &= \frac{\beta^2}{4}\left(\frac{y_0}{\bar{y}}\right)^\gamma \\
q_{HL}^0 &= \frac{\beta^2}{4}\left(\frac{y_0}{\underline{y}}\right)^\gamma \\
q_{LH}^0 &= \frac{\beta^2}{4}\left(\frac{y_0}{\bar{y}}\right)^\gamma \\
q_{LL}^0 &= \frac{\beta^2}{4}\left(\frac{y_0}{\underline{y}}\right)^\gamma
\end{aligned}$$

Note that – not incidentally – $q_{HH}^0 = q_{LH}^0$ and $q_{HL}^0 = q_{LL}^0$. What matters is only the state in the second period. This shows that one could simply trade (and re-trade) *long-run AD securities* that pay out always only in the high state or low state.

Next, in period 1, we need to distinguish between the state we are in. If the state is $H$, we have

$$\begin{aligned}
q_{HH}^1 &= \frac{\beta}{2}\left(\frac{c_H}{c_{HH}}\right)^\gamma = \frac{\beta}{2}\left(\frac{\bar{y}}{\bar{y}}\right)^\gamma = \frac{\beta}{2} \\
q_{HL}^1 &= \frac{\beta}{2}\left(\frac{c_H}{c_{HL}}\right)^\gamma = \frac{\beta}{2}\left(\frac{\bar{y}}{\underline{y}}\right)^\gamma \\
q_{LH}^1 &= q_{LL}^1 = 0
\end{aligned}$$

Equivalently, we have in state $L$

$$\begin{aligned}
q_{LL}^1 &= \frac{\beta}{2} \\
q_{LH}^1 &= \frac{\beta}{2}\left(\frac{\underline{y}}{\bar{y}}\right)^\gamma \\
q_{HL}^1 &= q_{HH}^1 = 0.
\end{aligned}$$

2. The bond that matures at $t = 1$ has the price

$$q_{b,1}^0 = q_H^0 + q_L^0 = \frac{\beta}{2}\left[\left(\frac{y_0}{\bar{y}}\right)^\gamma + \left(\frac{y_0}{\underline{y}}\right)^\gamma\right]$$

The bond that matures at period 2 has a state-contingent price in period $t = 1$. Why? Its price depends on the income the household receives in period 1.

Hence, we have

$$q_{b,2}^H = q_{HH}^1 + q_{HL}^1 = \frac{\beta}{2}\left[1 + \left(\frac{\bar{y}}{\underline{y}}\right)^\gamma\right]$$
$$q_{b,2}^L = q_{LH}^1 + q_{LL}^1 = \frac{\beta}{2}\left[1 + \left(\frac{\underline{y}}{\bar{y}}\right)^\gamma\right]$$

To obtain the price of the bond in period 0, we need to find a portfolio that replicates the payoff in period 1. Hence, we have

$$q_{b,2}^0 = q_H^0 \frac{\beta}{2}\left[1 + \left(\frac{\bar{y}}{\underline{y}}\right)^\gamma\right] + q_L^0 \frac{\beta}{2}\left[1 + \left(\frac{\underline{y}}{\bar{y}}\right)^\gamma\right] = \frac{\beta^2}{2}\left[\left(\frac{y_0}{\bar{y}}\right)^\gamma + \left(\frac{y_0}{\underline{y}}\right)^\gamma\right]$$

Not by coincidence(!), this is equal to $q_{HH}^0 + q_{HL}^0 + q_{LH}^0 + q_{LL}^0$.

Remark: In general, we can work backwards when pricing assets when market are complete (think of a full set of AD securities so that we can achieve any payoff). This works as the Euler equation just substitutes out for different MRS and probabilities across time. This should be evident from the FOC and from the algebra when solving the question correctly. The prices of the AD securities simply "pick up" the right MRS state-by-state.

3. We have now a long-lived asset. This asset pays out a dividend in period 1 and 2 that is state-dependent. Hence, the tree's payoffs are given by $d_1 + p_1$ in the first period and by $d_2$ in the second period. Note that the tree is traded *ex dividend*, i.e., without the fruit attached.

Now, we work backwards again. The price of the tree in period 1 is given by

$$p_\ell(H) = q_{HH}^1 \cdot 1 + q_{HL}^1 \cdot 2 = \frac{\beta}{2} + \beta\left(\frac{\bar{y}}{\underline{y}}\right)^\gamma$$
$$p_\ell(L) = q_{LH}^1 \cdot 1 + q_{LL}^1 \cdot 2 = \frac{\beta}{2}\left(\frac{\underline{y}}{\bar{y}}\right)^\gamma + \beta$$

The price in period 0 is then given by

$$p_\ell(0) = q_H^0(d_H + p_\ell(H)) + q_L^0(d_L + p_\ell(L)).$$

Plugging in the appropriate values, we get

$$p_\ell(0) = q_H^0(1 + p_\ell(H)) + q_L^0(2 + p_\ell(L)) = \left(\frac{\beta}{2} + \frac{\beta^2}{2}\right)\left[\left(\frac{y_0}{\bar{y}}\right)^\gamma + 2\left(\frac{y_0}{\underline{y}}\right)^\gamma\right]$$

<u>Remark:</u> This looks like discounted cash-flow pricing. The first term acts like a discount factor. The second term is expressing the dividend weighted by a risk factor that is given by the MRS between period 0 consumption and consumption in the low or high state in any period.

4. To price the orange tree, we just apply the formulas above, since the only difference is given by the dividend process. We optain for the period 1 prices

$$p_o(H) = q_{HH}^1 \cdot 2 + q_{HL}^1 \cdot 1 = \beta + \frac{\beta}{2}\left(\frac{\bar{y}}{\underline{y}}\right)^\gamma$$

$$p_o(L) = q_{LH}^1 \cdot 2 + q_{LL}^1 \cdot 1 = \beta\left(\frac{\underline{y}}{\bar{y}}\right)^\gamma + \frac{\beta}{2}$$

The price in period 0 needs to be

$$p_o(0) = \left(\frac{\beta}{2} + \frac{\beta^2}{2}\right)\left[2\left(\frac{y_0}{\bar{y}}\right)^\gamma + \left(\frac{y_0}{\underline{y}}\right)^\gamma\right]$$

5. We first compare the two bonds. Their prices are given by

$$q_{b,1}^0 = \frac{\beta}{2}\left[\left(\frac{y_0}{\bar{y}}\right)^\gamma + \left(\frac{y_0}{\underline{y}}\right)^\gamma\right]$$

$$q_{b,2}^0 = \frac{\beta^2}{2}\left[\left(\frac{y_0}{\bar{y}}\right)^\gamma + \left(\frac{y_0}{\underline{y}}\right)^\gamma\right]$$

The only difference between the two bonds is the price of time, $\beta$. This is intuitive. Both bonds pay out 1 unit of consumption in every state when they mature. Hence, buying and holding the bond only yields a difference in consumption across time. Interestingly, the second bond is *not* risk-free when bought or sold in period 1. Hence, there is intertemporal risk with respect to the risk-free rate which is given by the fact that $q_{b,2}^H \neq q_{b,2}^L$.

Next, we compare the prices for the two trees. We have that

$$\frac{p_\ell - p_o}{\frac{\beta}{2} + \frac{\beta^2}{2}} = \left(\frac{y_0}{\underline{y}}\right)^\gamma - \left(\frac{y_0}{\bar{y}}\right)^\gamma > 0$$

Why is the lemon tree worth more? It pays out more as a dividend in the low state than in the high state. Hence, it insures against consumption risk and is a better hedge against risk and, thus, more valuable.

One really cannot compare the risk-free bond and the two trees as they have a very different payoff structure.

**<u>WARNING:</u>** I flipped the lemon and the orange tree. You can just flip the answers.

6. We first find the conditional risk-free rate in period 0. This is just given by the risk-free bond we found earlier. Hence, we have that

$$\frac{1}{1+r_f(H)} = q_b^0(H) = \frac{\beta}{2}\left[\left(\frac{\bar{y}}{\bar{y}}\right)^\gamma + \left(\frac{\bar{y}}{\underline{y}}\right)^\gamma\right]$$
$$\frac{1}{1+r_f(L)} = q_b^0(L) = \frac{\beta}{2}\left[\left(\frac{\underline{y}}{\bar{y}}\right)^\gamma + \left(\frac{\underline{y}}{\underline{y}}\right)^\gamma\right]$$

Since the two states happen with probability 1/2, the unconditional risk-free rate in period 0 is given by

$$\frac{1}{2}\left[(1+r_f(H)) + (1+r_f(L))\right] = \frac{1}{\beta}$$

Next, we are calculating the conditonal return on the orange tree (equity) with pays the endowment as the dividend. Note that uncertainty is iid. Hence, the expected payoff in $t = 1$ is *independent* of the state in $t = 0$. Thus,

$$E[p_1 + d_1] = \frac{1}{2}(p_1(H) + p_1(L)) + \frac{3}{2}$$

The price in period 0 depends on the price in the current state. Hence, we have (ex dividend) for the expected return in period 0

$$(1 + E[r_e(H)]) = \frac{E[p_1 + d_1]}{p_0(H)} = \frac{\frac{1}{2}(2+2^\gamma) + ((1/2)^\gamma + 1/2) + 3/\beta}{(1+\beta)(2+2^\gamma)}$$
$$(1 + E[r_e(L)]) = \frac{E[p_1 + d_1]}{p_0(L)} = \frac{\frac{1}{2}(2+2^\gamma) + ((1/2)^\gamma + 1/2) + 3/\beta}{(1+\beta)2(1/2+(1/2)^\gamma)}$$

Combining these two terms we obtain for unconditional return on equity

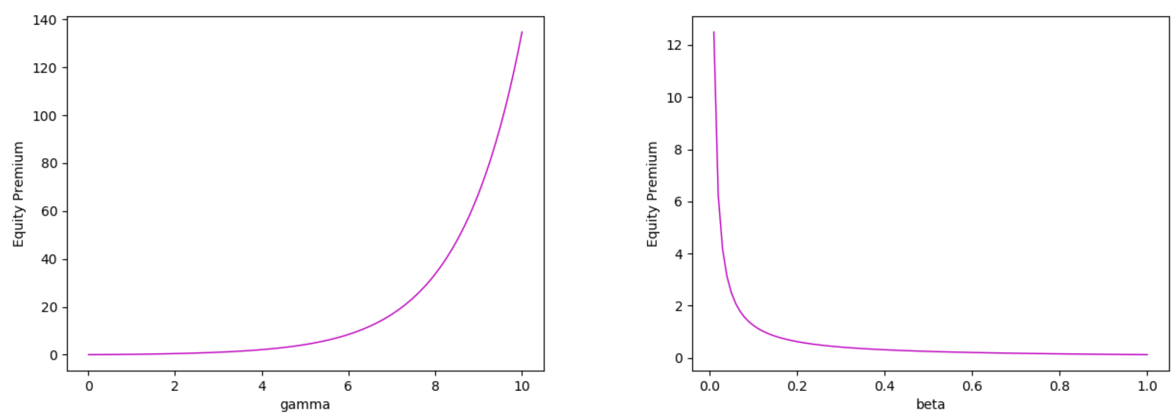
$$(1 + E[r_e(0)]) = \frac{1}{2}\left(\frac{\frac{1}{2}(2 + 2^{\gamma}) + ((1/2)^{\gamma} + 1/2) + 3/\beta}{1 + \beta}\right)\left(\frac{1}{(2 + 2^{\gamma})} + \frac{1}{2(1/2 + (1/2)^{\gamma})}\right) = \Upsilon$$

The unconditional equity premium is the difference between the two expected returns or

$$E[r_e(0)] - r_f = \Upsilon - \frac{1}{\beta}$$

Unfortunately, it is a somewhat ugly expression.

<u>Remark:</u> The graphs below show some comparative statics of the equity premium with respect to the coefficient of relative risk aversion $\gamma$ and $\beta$.




Increasing $\gamma$ and decreasing $\beta$ generates larger equity premiums. Using standard parameter values and data on equity returns and the risk-free rate, however, yield an equity premium that is too low compared to the data and a risk-free rate that is too high. Hence, one needs to change our basic asset pricing model to fit the data. One way to do that is to change equity risk. Another one is to decrease the risk-free rate through modelling credit market imperfections.

7. A call option gives the right to buy the tree at the strike price. Since the strike price is the price of the tree in the low state, the payoff structure for the option is given by

$$C_1(H) = \max\{p_o(H) - p_o(L), 0\}$$
$$C_1(L) = \max\{p_o(L) - p_o(L), 0\} = 0$$

Hence, we have that

$$C(H) = \beta \left( \frac{\bar{y}^\gamma - \underline{y}^\gamma}{\bar{y}^\gamma} \right) + \frac{\beta}{2} \left( \frac{\bar{y}^\gamma - \underline{y}^\gamma}{\underline{y}^\gamma} \right) > 0$$

Using the price for the AD security that pays out one unit of consumption in period 1 in state $H$, the price of the call in period 0 is given by

$$\begin{aligned} C_0 &= q^0(H) \cdot C(H) \\ &= \left( \frac{y_0}{\bar{y}} \right)^\gamma \left[ \frac{\beta^2}{2} \left( \frac{\bar{y}^\gamma - \underline{y}^\gamma}{\bar{y}^\gamma} \right) + \frac{\beta^2}{4} \left( \frac{\bar{y}^\gamma - \underline{y}^\gamma}{\underline{y}^\gamma} \right) \right] \\ &= \frac{\beta^2}{4} \left[ \left( \frac{y_0}{\bar{y}} \right)^\gamma + \left( \frac{y_0}{\underline{y}} \right)^\gamma - 2 \left( \frac{y_0 \underline{y}}{\bar{y}^2} \right)^\gamma \right] \end{aligned}$$

Remark: An option is just a state-contingent claim and can be replicated (and priced) via AD securities.

**Answer to Question 4:**

1. The government's budget constraints are given by

$$g_1 = \tau_1 w_1 n_1$$

$$g_2 = \tau_2 w_2 n_2$$

$$g = g_1 + g_2$$

   Note that the absence of borrowing and lending does not allow the government to run a (temporary) deficit or surplus. However, by varying $\tau_1$ vs. $\tau_2$ – and, henceforth, $g_1$ and $g_2$ – it can shift the tax burden across periods.

2. Households take the wage $w$, profits $(\Pi_1, \Pi_2)$, as well as government policy $(g, \tau_1, \tau_2)$ as given and solve the following problem

$$\max_{c_1, c_2, n_1, n_2} \frac{c_1^{1-\gamma}}{1-\gamma} + \theta \frac{(1-n_1)^{1-\eta}}{1-\eta} + \frac{c_2^{1-\gamma}}{1-\gamma} + \theta \frac{(1-n_2)^{1-\eta}}{1-\eta}$$

   subject to

$$c_1 \leq (1 - \tau_1) w_1 n_1 + \Pi_1$$

$$c_2 \leq (1 - \tau_2) w_2 n_2 + \Pi_2$$

The firm takes wages as given and solves

$$\max_{n_t} n_t^\alpha - w_t n_t$$

A *competitive equilibrium* for a government policy $(g, \tau_1, \tau_2)$ is then given by prices $w_t$ and an allocation $(c_1, c_2, n_1, n_2)$ such that:

- households maximize utility taking the policy, profits and prices as given
- firms maximize profits taking wages as given
- markets clear, i.e.

$$c_1 + g_1 = n_1^\alpha$$
$$c_2 + g_2 = n_2^\alpha$$

Note that we have not put any restrictions on government policy. This implies that for values of $g$ that are too high, there might not exist any equilibrium. For the remainder of the question, we will restrict ourselves to feasible gov't policies where an equilibrium exists.

3. The FOC for the firm's problem is given by

$$w_t = \alpha n_t^{\alpha-1}$$

which is independent of taxes.

The FOCs for the household's problem are

$$\frac{(1-n_1)^\eta}{c_1^\gamma} = \frac{\theta}{(1-\tau_1)w_1}$$
$$\frac{(1-n_2)^\eta}{c_2^\gamma} = \frac{\theta}{(1-\tau_2)w_2}$$

for the first and the second period.

Remark: For simplicity, there is no direct intertemporal choice for the household. One could still allow for borrowing and lending and derive an intertemporal Euler equation which is given by

$$\left(\frac{c_2}{c_1}\right)^\gamma = (1+r).$$

This equation would simply pin down the interest rate as a function of taxes $\tau_1$ and $\tau_2$. Since the government cannot borrow or lend and there is a representative household, there cannot be any savings or borrowing. Hence, the problem in each period can be viewed separately – except for the government's constraint of having to raise enough taxes for a total of $g$.

4. Since $\alpha = 1$, the production function is linear so that

$$w_1 = w_2 = 1.$$

This also implies that profits are zero because of constant returns-to-scale in production. Using the budget constraint in the first-order condition of the household, we obtain

$$\frac{1 - n_t}{(1 - \tau_t) n_t w_t} = \frac{\theta}{1 - \tau_t}$$

so that

$$n_t = \frac{1}{1 + \theta}.$$

From the budget constraint, we then have that consumption is given by

$$c_t = \frac{1 - \tau_t}{1 + \theta}$$

The government budget constraint is given by

$$g_t = \tau_t n_t = \frac{\tau_t}{1 + \theta}$$

so that we have in equilibrium

$$c_t = \frac{1 - g_t(1 + \theta)}{1 + \theta}.$$

The solution yields the following insights. First, labour supply $n$ is *independent* of the tax rate $\tau$ and, consequently, total output does not depend on it. This is a consequence of log utility where income and substitution effects cancel out exactly. Higher taxes (or gov't consumption) implies a fall in income and a fall in (effective) wages relative to the normalized price of consumption. Hence, there is a negative income effect, but a positive substitution effect for the labour supply choice; however, the two effects exactly cancel each other. Second, wages do not depend on labour supply, since the marginal product of labour is constant and normalized to 1. Third, consumption reacts 1-1 to changes in taxes over time.

5. The welfare maximizing policy given a total expenditure of g across periods is given by $\tau_1 = \tau_2$ or, equivalently $g_1 = g_2$. This is simply a consequence of tax smoothing across periods. The intuition is that it is optimal to smooth distortions across time.

   To verify this intuition, we can use the fact $g_2 = g - g_1$ and the equilibrium conditions from part (d) to obtain an indirect utility function for the household that is given by

$$\log\big(1 - g_1(1+\theta)\big) + \log\big(1 - (g - g_1)(1+\theta)\big)$$

   where he have neglected all constant terms. Differentiating with respect to $g_1$ we obtain – again neglecting multiplying constants

$$g - 2g_1 = 0$$

   as a first-order condition which confirms our intuition.

   Remark: In general, one would need to solve a so-called *Ramsey problem*. A social planner chooses the tax policy to maximize the welfare of the household taking into account as a constraint (!) that taxes lead to a particular equilibrium in the economy. For the case above, the constraint has been used directly in the indirect utility function which is the objective of the Ramsey planner.

6. Note first that the problems in period 1 and 2 are identical. The only difference is the level of government spending $g_t$. Hence, we can use the household's first-order condition, the expression for the wage rate and the budget constraint to define $n$ as a function of

$\tau$,

$$\frac{(1-n)^\eta}{c^\gamma} = \frac{\theta}{(1-\tau)w}$$
$$\frac{(1-n)^\eta}{(n^\alpha - \tau w n)^\gamma} = \frac{\theta}{(1-\tau)\alpha n^{\alpha-1}}$$
$$\frac{\theta(n^\alpha - \tau\alpha n^\alpha)^\gamma}{(1-n)^\eta(1-\tau)\alpha n^{\alpha-1}} - 1 = 0$$

where the second step uses the fact that the consumer owns the firm and obtains its profits. This is equivalent – in equilibrium – to obtain all the output minus the taxes paid on labour income.

Hence, computing all the variables boils down to solving again one non-linear equation in one unknown $n$ for all possible values of $n$. Note that this (implicitly) pins down the solution for all (feasible) gov't expenditures in period $t$. A sample program in Matlab looks like this:[1]

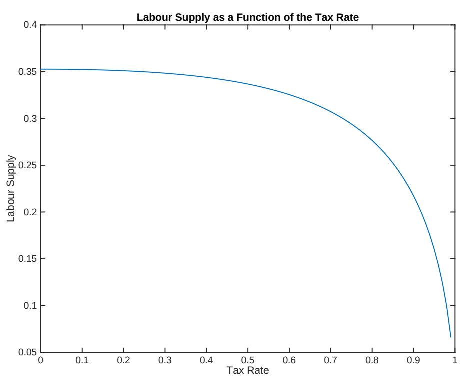
```
clear all

tau=0:0.01:0.99;
for i=1:length(tau);
   taui=tau(i);
   n_star = fsolve(@(n) foc(n,taui),0.5);
   nstar(i) = n_star(1);
end;

cstar = (nstar.^.5).*(1 - tau/2)
wstar = 1./(2*nstar.^.5)
ustar = cstar.^(1-2)/(1-2) + (1-nstar).^(1-2)/(1-2)

plot(tau,nstar)
plot(tau,cstar)
```

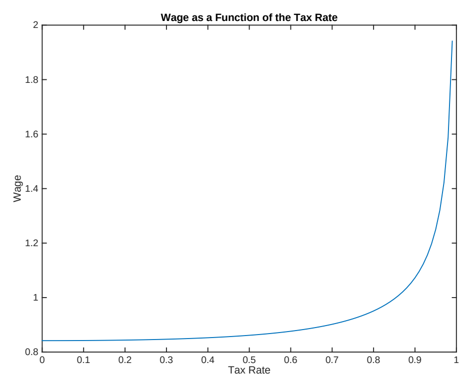
[1]Frederic Tremblay provided the code for solving the problem.

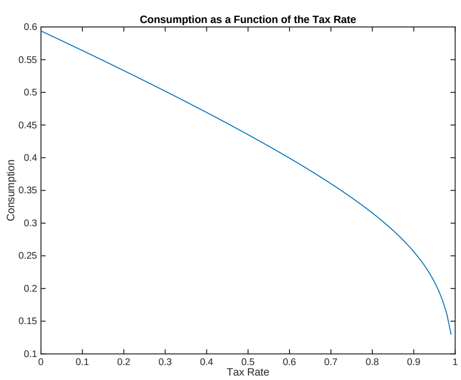
```
plot(tau,wstar)
plot(tau,ustar)
```

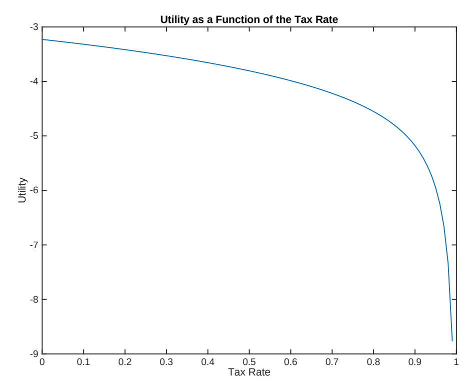
First, one defines a vector for all possible values of $\tau$. Then, one runs a loop over all these values. The loop involves two steps. First, it solves the non-linear equation using `fsolve`. Second, it stores the result in a vector for the equilibrium $n$. The remainder of the program simply calculates all other variables as a function of $n$ and plot these against $\tau$.

The graphs below show labour supply, consumption, wages and utility as a function of tax rates in any period.



Labour supply is falling as a function of taxes due to distortions in after tax wages, while wages themselves increase with taxes. The reason for the latter result is that the marginal product of labour increases with lower labour input in production. Consumption decreases as total income decreases with taxes. Overall utility is also decreasing in taxes.

7. The government revenue in each period $g_t$ is given by

$$g_t = \tau_t w_t n_t$$

From part (e), we know that utility is maximized by setting $g_1 = g_2$ because the distortions are minimized. Hence, total output is also maximized by equal taxation across period and, consequently, doing so maximizes tax revenues. Since the problem is symmetric in both periods, we can express the total revenue $g$ using the equation above as

$$\frac{g}{2} = \tau w n$$

or

$$g = \tau \sqrt{n}$$

Using the equilibrium labour supply $n^*(\tau)$ computed in part (f) we obtain that we have to solve

$$\max_{\tau} \tau \sqrt{n^*(\tau)}$$

We can adjust the computer code for part (f) to obtain the revenue maximizing tax as follows[2]

```
g = tau.*nstar.^.5;
[gmax, imax] = max(g);
taustar = tau(imax);
```

[2]Frederic Tremblay provided the code for solving the problem.

The graph below shows government revenue as a function of a flat tax $\tau$ across periods. The figure exhibits a so-called "Laffer curve" where at first revenues are increasing in the tax rate, but as the tax rate grows they start declining. The reason is that the tax base (labour supply) decreases with higher tax rates – eventually overpowering the effect of higher rates on revenue. Government revenue is maximized at $\tau \approx 0.85$ with $g \approx 0.4270$ and $n \approx$