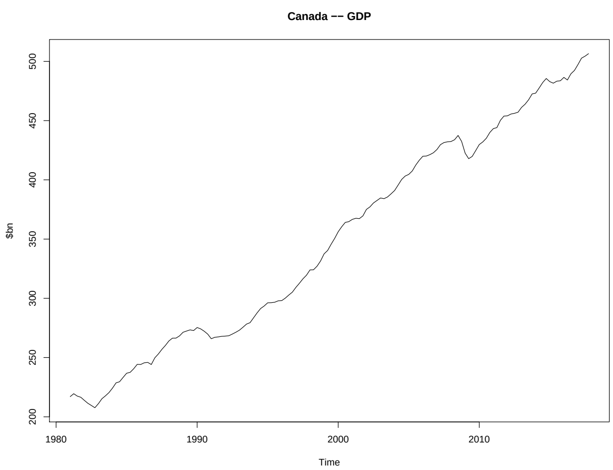
Economics 815 **Winter 2019**

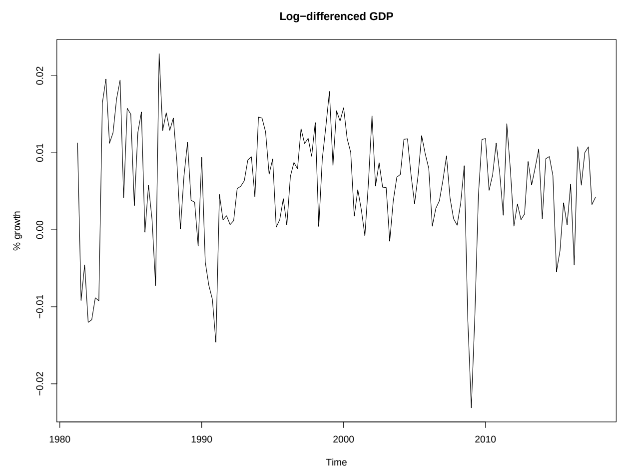
Macroeconomic Theory **Thorsten Koeppl**

Answer Key for Assignment 1

<u>Answer to Question 1:</u>

The first two graphs show raw Canadian GDP and detrended GDP by log-first differences (growth rates).





The next three graphs below show Canadian GDP filtered for different values of the Hodrick-Prescott Filter. The first one gives a linear trend, while the other two are very similar, both in trends and cycle.

The next two graphs show the periodogram of the original data series and for the log-differenced data (growth rates). Note that the scale on the x-axis is given by frequencies that range from 0 to 2π and, consequently, is measured in radians.

We clearly see that the raw data pick up the lowest frequency which is the trend, while the log-differenced data have very much a flat, uniform distribuiton indicating weight on all frequencies.

For the HP filtered data, we only plot the periodograms for $\lambda = 1600$. The plot on the left side shows the trend component, while the plot on the right side shows the cyclical component. Note that the x-axis has now been converted into "frequency" in terms of the number of

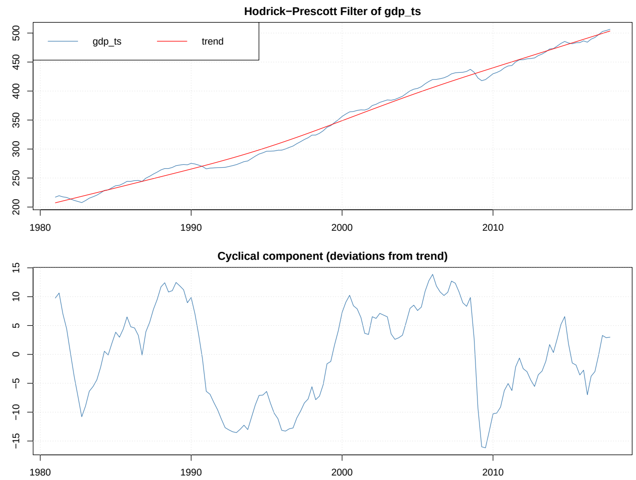


Figure 1: Linear Trend – $\lambda = 100000$

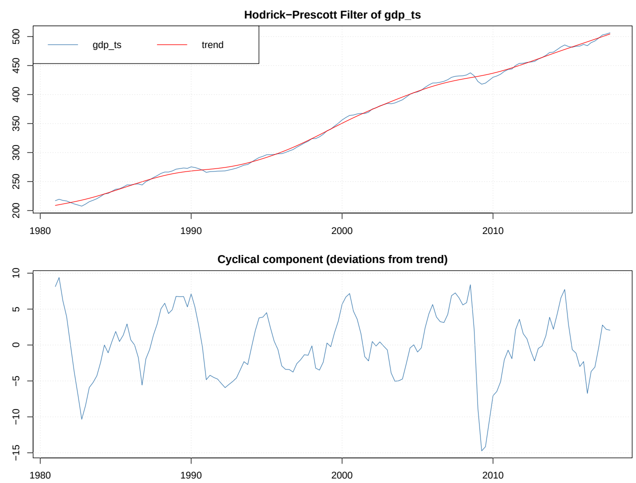


Figure 2: Business Cycle Frequency – $\lambda = 1600$

cycles in the period. Hence, 1 means the lowest frequency ("one cycle") and 75 means the highest frequency ("two quarter cycle").

The HP filter emphasizes fequencies that are in the range of 4 and 6 cycles during the last roughly 40 years. This is sometimes not helpful. To compare the cyclical component between $\lambda = 1600$ and $\lambda = 400$, we convert the x-axis again to reflect the length of a cycle or the "period" which is simply given by the reciprocal of the frequency. Note that we rescale the x-axis to reflect the number of quarters for a cycle (i.e., we multiply by 148).

With $\lambda = 1600$, we pick up the length of 30 quarters as the most prominent frequency. This

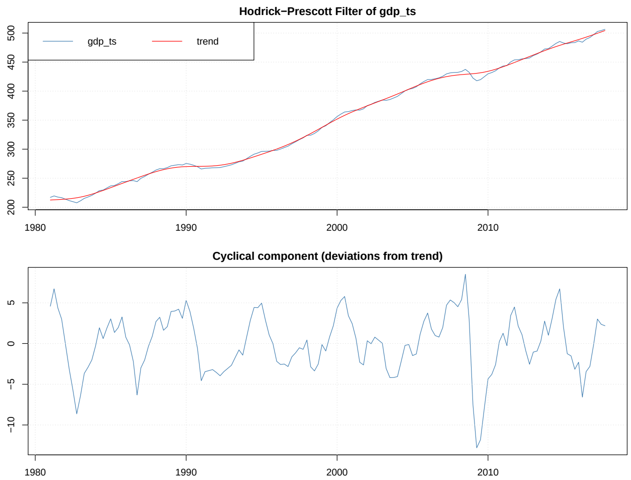
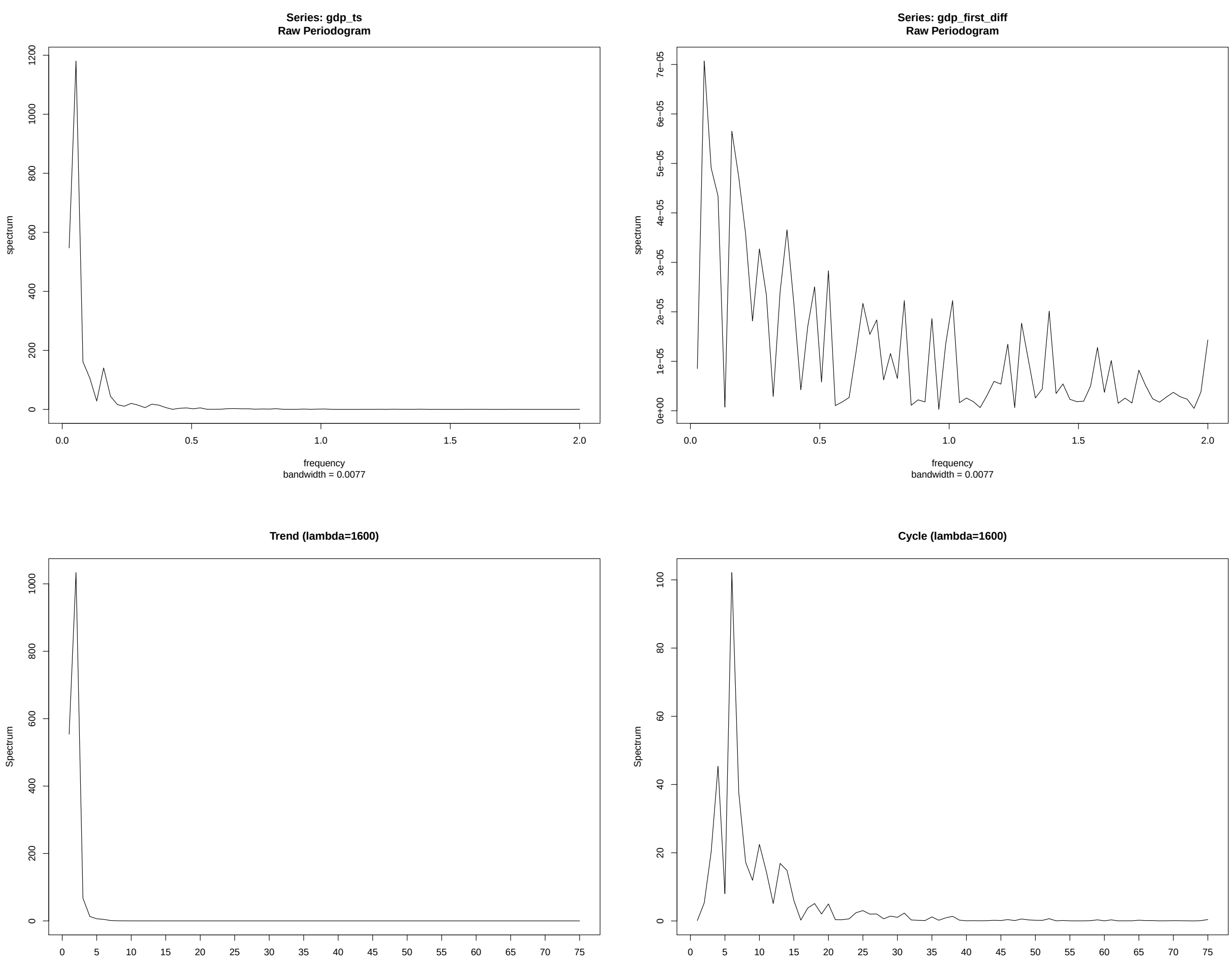
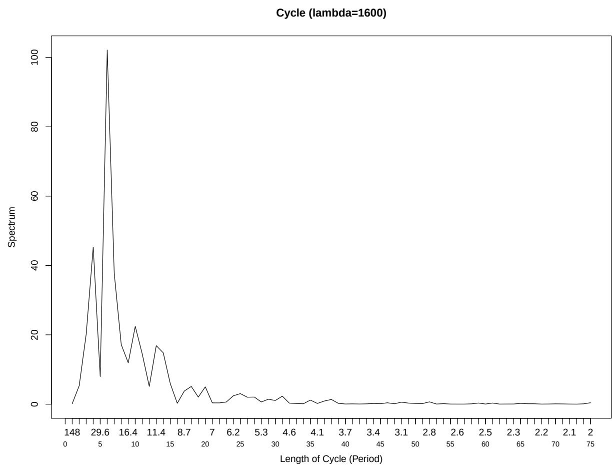
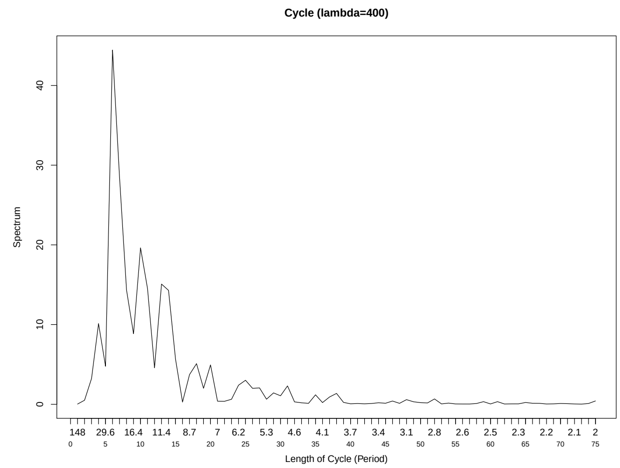


Figure 3: Higher Frequency – $\lambda = 400$







points to about 8 years for a cycle. Reducing the smoothing parameter for the trend to $\lambda = 400$ we should give more weight to shorter periods or higher frequency movements. We see two effects. Cycles now appear most prominent at 21 quarters or about 5 years with more weight given to even shorter cycles. Bottomline: detrending and smoothing matters!

Answer to Question 2:

1. Each household i solves the following problem

$$\max_{c_1^i, c_2^i} \ln c_1^i + \beta_i \ln c_2^i$$

subject to

$$c_1^i + s^i \leq y_1^i$$
$$c_2^i = (1+r)s^i + y_2^i.$$

We can then derive an intertemporal budget constraint that is given by

$$c_1^i + \frac{c_2^i}{1+r} \leq y_1^i + \frac{y_2^i}{1+r}$$

Taking a first-order condition, we obtain for both households

$$\frac{c_2^i}{\beta_i c_1^i} = (1+r).$$

Using this in the budget constraint we obtain

$$c_1^i + \frac{\beta c_1^i(1+r)}{1+r} = y_1^i + \frac{y_2^i}{1+r}$$

so that household i's demand for consumption in the first period and second period is being equal to

$$c_1^i = \left(\frac{1}{1+\beta}\right)\left[y_1^i + \frac{y_2^i}{1+r}\right]$$
$$c_2^i = \left(\frac{\beta}{1+\beta}\right)\left[(1+r)y_1^i + y_2^i\right].$$

Then, we have from market clearing that

$$\sum_i c_t^i = \sum_i y_t^i = 1$$

for both periods t. Using the expression for c_t^i, we obtain that the equilibrium interest rates is given by

$$\frac{1}{\beta} = 1 + r.$$

Hence, we have that

$$c_1^1 = c_2^1 = \frac{1}{1+\beta}$$
$$c_1^2 = c_2^2 = \frac{\beta}{1+\beta}.$$

Both households have a flat consumption profile; i.e., they fully insure each other against the fluctuation in income. This is done by the second household borrowing an amount

$$s^2 = 1 - \left(\frac{1}{1+\beta}\right).$$

The level of consumption for both households is, however, different. A household's net present wealth depends on when he receives his endowment.

2. When $\beta = 1$, there is no discounting. Hence, when households receive their endowment does not matter. Consequently, $r = 0$, and $c_1^1 = c_2^1 = c_1^2 = c_2^2 = \frac{1}{2}$. That is, households share consumption equally among themselves and across periods.

   When $\beta \to \infty$, households value consumption tomorrow infinitely more than consumption today. In this case, $1 + r \to 0$, and $c_1^1 = c_2^1 \to 0$, while $c_1^2 = c_2^2 \to 1$. Receiving one's endowment early becomes useless.

When $\beta \to 0$, households only value current consumption. In this case, $1 + r \to \infty$, and $c_1^1 = c_2^1 \to 1$, $c_1^2 = c_2^2 \to 0$. Household 1 is not willing to lend to household 2 at any positive interest rate.

Answer to Question 3:

We define the gain in utility from a payoff as

$$\Delta = u(1) - u(0)$$

The lottery is a single, initial coin flip that determines whether consumption occurs before or after the odd period. Comparing the two utility gains, we obtain

$$\frac{1}{2}\sum_{t=1}^{\infty}\beta^{2t-1}\Delta + \frac{1}{2}\sum_{t=1}^{\infty}\beta^{2t+1}\Delta \lesseqgtr \sum_{t=1}^{\infty}\beta^{2t}\Delta$$

Hence,

$$\begin{aligned}
\sum_{t=1}^{\infty}\frac{1}{2}\beta^{2t-1} + \sum_{t=1}^{\infty}\frac{1}{2}\beta^{2t+1} \quad &\lesseqgtr \quad \sum_{t=1}^{\infty}\beta^{2t} \\
\frac{1}{2}\left(\frac{1}{\beta} + \beta\right)\cdot\left(\sum_{t=1}^{\infty}\beta^{2t}\right) \quad &\lesseqgtr \quad \sum_{t=1}^{\infty}\beta^{2t} \\
(\beta - 1)^2 \quad &\lesseqgtr \quad 0
\end{aligned}$$

Therefore, the person prefers lottery (ii) over lottery (i) for any $\beta \in (0,1)$. People prefer gambles over time rather than certain spikes in consumption in odd periods.

<u>Remark:</u> This follows straight from Jensen's inequality. We can consider this to be a sequence of payoffs that compare $1/2\beta^{t-1} + 1/2\beta^{t+1}$ to β^t. The function $f(t) = \beta^t$ is strictly convex. Hence, we get

$$E[\beta^t] > \beta^{E[t]}$$

<u>Remark:</u> We could also consider a sequence of coin flips taking place every even period to determine immediate consumption of 1 or delayed consumption of 1. This would yield the

following distribution of utility every even period for $t > 1$

$$u(2) \text{ with prob. } 1/4$$
$$u(1) \text{ with prob. } 1/2$$
$$u(0) \text{ with prob. } 1/4$$

The first period $t = 1$ has $u(1)$ and $u(0)$ with equal probability. The answer will then depend on $u(2) - u(0)$ as well as β. Try it!

Answer to Question 4:

1. Let a_t be the assets the consumer carries into period t. His budget constraint is then given by

$$c_t + a_{t+1} = y_t + (1+r)a_t$$

In period $t+1$, we have that in every state a similar budget constraint holds. Weighting by the probability of the state in $t+1$, we then obtain that

$$E_t[c_{t+1} + a_{t+2}] = E_t[y_{t+1} + (1+r)a_{t+1}]$$
$$E_t[c_{t+1} + a_{t+2}] = E_t[y_{t+1}] + (1+r)a_{t+1}$$

because $(1+r)a_{t+1}$ is a constant. Hence, we have that

$$(1+r)a_t = c_t - y_t + \left(\frac{1}{1+r}\right) E_t[c_{t+1} - y_{t+1} + a_{t+2}]$$

Iterating forward, using the law of iterated expectations and an appropriate limit condition, we obtain

$$\sum_{s=0}^{\infty}(1+r)^{-s}E_t[c_{t+s}] = (1+r)a_t + \sum_{s=0}^{\infty}(1+r)^{-s}E_t[y_{t+s}]$$

We can now use the Euler equation $E_t[c_{t+s}] = c_t$ to obtain

$$\frac{1+r}{r}c_t = (1+r)a_t + \sum_{s=0}^{\infty}(1+r)^{-s}E_t[y_{t+s}]$$
$$c_t = ra_t + \frac{r}{1+r}\sum_{s=0}^{\infty}(1+r)^{-s}E_t[y_{t+s}]$$

The AR(1) income process yields through repeated substitution, applying the law of iterated expectations and using the fact that $E_t[\epsilon_{t+1}] = 0$,

$$E_t[y_{t+s}] = \rho^s y_t$$

so that

$$\sum_{s=0}^{\infty}(1+r)^{-s}E_t[y_{t+s}] = y_t \sum_{s=0}^{\infty}\left(\frac{\rho}{1+r}\right)^s = y_t\left(\frac{1+r}{1+r-\rho}\right)$$

Combining, we obtain for the consumption function

$$c_t = ra_t + \left(\frac{r}{1+r-\rho}\right)y_t$$

Consumption is given by a fraction of wealth plus a fraction of current income.

2. The consumption function in the previous period is given by

$$c_{t-1} = ra_{t-1} + \left(\frac{r}{1+r-\rho}\right)y_{t-1}$$

We now use the budget constraint in period $t-1$

$$c_{t-1} + a_t = y_{t-1} + (1+r)a_{t-1}$$

to obtain

$$c_{t-1} = \left(\frac{r}{1+r}\right)(c_{t-1} + a_t - y_{t-1}) + \left(\frac{r}{1+r-\rho}\right)y_{t-1}$$

or

$$c_{t-1} = ra_t + \left(\frac{r}{1+r-\rho}\right)\rho y_{t-1}$$

This allows us to substitute for ra_t in the consumption function to obtain

$$c_t = c_{t-1} + \left(\frac{r}{1+r-\rho}\right)(y_t - \rho y_{t-1}) = c_{t-1} + \left(\frac{r}{1+r-\rho}\right)\epsilon_t$$

3. Note that both $c_t - c_{t-1}$ and ϵ_t are random variables. Hence, we have

$$V(c_t - c_{t-1}) = \left(\frac{r}{1+r-\rho}\right)^2 V(\epsilon_t)$$

If $\rho \to 0$, all income shocks are temporary and the income shock follows a random walk. Consumption growth would then be least volatile.

If $\rho \to 1$, all income shocks are permanent. Hence, consumption would react 1-1 to the shock. Consumption growth would be most volatile.

One could estimate r and ρ from the data. A common result is that consumption *varies less* than what this model implies. This puzzle is often referred to as “excess smoothness of consumption”.