ECON 815

Long-run Growth

Winter 2020

Balanced Growth Path

Definiton: An economy is on a balanced growth path if all variables grow at the same rate.

We have some well established facts that we would like to capture.

- Y/L and K/L grow over time.
- K/Y is constant.
- Wages increase over time
- Interest rates remain constant.
- Income shares remain constant over time.

One can easily verify that the neoclassical production function in a balanced growth part is consistent with the stylized growth facts.

Some Accounting

Dynamic evolution of the economy is given by feasibility:

$$Y = C + X = C + I - \delta K$$

Output given by some production function

$$Y = F(K, L)$$

Fix L and assume that the MPK is decreasing. If $C = (1 - s)Y$, we converge to a steady state (Solow).

More generally, we have

$$g_Y = \frac{K}{Y}\frac{\partial F}{\partial K} g_K + \frac{L}{Y}\frac{\partial F}{\partial L} g_L = \eta_K g_K + \eta_L g_L$$

Source of Growth with BGP?

In a BGP the capital-output ratio $\frac{K}{Y}$ remains constant. Then, we must have $g_Y = g_K = g$.

Hence:

$$g = \frac{\eta_L}{(1-\eta_K)} g_L$$

Implications:

1) There must be some source of exogenous growth for the economy to evolve along a BGP with positive growth.

2) This source affects effective labour – or equivalently – labour productivity (*labour-augmenting* technological progress).

3) Constant-returns-to-scale is a special case, since we have $\eta_K + \eta_L = 1$ and, thus, $g = g_Y = g_K = g_L$. Growth can come from any factor in a BGP.

BGP and Detrending

Assumption: No value of leisure

Growth is labour-augmenting, i.e. $N_t = N_0(1+\gamma)^t$, where $N_0 = 1$.

Detrending, we obtain per capita variables

$$y_t = \frac{Y_t}{(1+\gamma)^t}$$
$$c_t = \frac{C_t}{(1+\gamma)^t}$$
$$k_t = \frac{K_t}{(1+\gamma)^t}$$

Then, we are dealing with a stationary economy that is described by

$$y_t = k_t^\alpha$$
$$y_t = c_t + (1+\gamma)k_{t+1} - (1-\delta)k_t$$
$$u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}(1+\gamma)^{t(1-\sigma)}$$

Social planning problem

$$\max_{c_t, k_t} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} (1+\gamma)^{t(1-\sigma)}$$

subject to

$$k_t^\alpha = c_t + (1+\gamma)k_{t+1} - (1-\delta)k_t$$

Solution:

$$\frac{1}{\beta}\left(\frac{c_t}{c_{t+1}}\right)^{-\sigma}(1+\gamma)^\sigma = (1-\delta) + \alpha k_{t+1}^{\alpha-1}$$

This Euler equation governs the transition to a BGP that is given by

$$\alpha \bar{k}^{\alpha-1} = \frac{1}{\beta}(1+\gamma)^\sigma - (1-\delta)$$

What governs long-run interest rates?

Suppose we have

$$
\begin{aligned}
Y_t &= A_t K_t^{\alpha} N_t^{1-\alpha} \\
&= A_0 K_t^{\alpha} \left[(1+\mu)^{\frac{t}{1-\alpha}} (1+n)^t N_0 \right]^{1-\alpha} \\
&= K_t^{\alpha} (1+\gamma)^{t(1-\alpha)}
\end{aligned}
$$

where $A_0 = N_0 = 1$.

The (exogenous) growth rate is approximately given by

$$\gamma \simeq \frac{\mu}{1-\alpha} + n$$

In the BGP, interest rates must be constant and are given by

$$r = \alpha \bar{k}^{\alpha-1}$$

$$
\begin{aligned}
r &= \frac{(1+\gamma)^{\sigma}}{\beta} - (1-\delta) \\
&\simeq -\ln\beta + \sigma\gamma + \delta \\
&\simeq \theta + \delta + \sigma\left(\frac{\mu}{1-\alpha} + n\right)
\end{aligned}
$$

For nominal interest rates that we mostly observe, we can use the Fisher equation which is given by

$$1 + r = \frac{1+i}{1+\pi^{e}}$$

or

$$i \simeq r + \pi^{e}$$

where π^{e} is expected inflation.

What about risk?

Assume that expected growth is log-normally distributed with mean $E[\gamma]$ and variance $Var[\gamma]$. We then have that

$$E\left[(1+\gamma)^{-\sigma}\right] = e^{-\sigma E[\gamma]+\frac{1}{2}\sigma^2 Var[\gamma]}$$

From the Euler condition, we obtain for a risk-free investement

$$1 = \beta E\left[(1+\gamma)^{-\sigma}(1+r_f)\right]$$

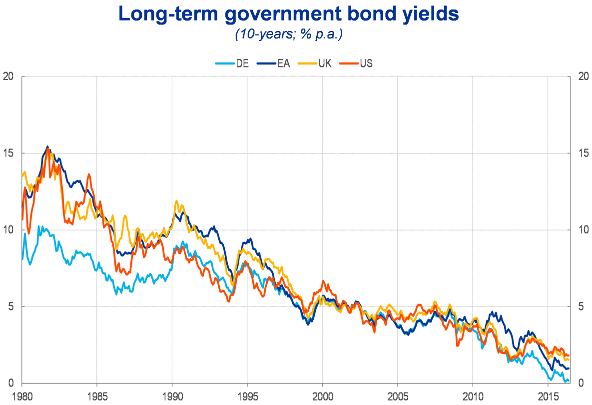
Taking logs, we get

$$r_f = \underbrace{\theta}_{\text{time}} + \sigma \underbrace{E[\gamma]}_{\text{growth}} - \frac{1}{2}\sigma^2 \underbrace{Var[\gamma]}_{\text{risk}}$$

which gives us a theory of the long-run risk-free rate.

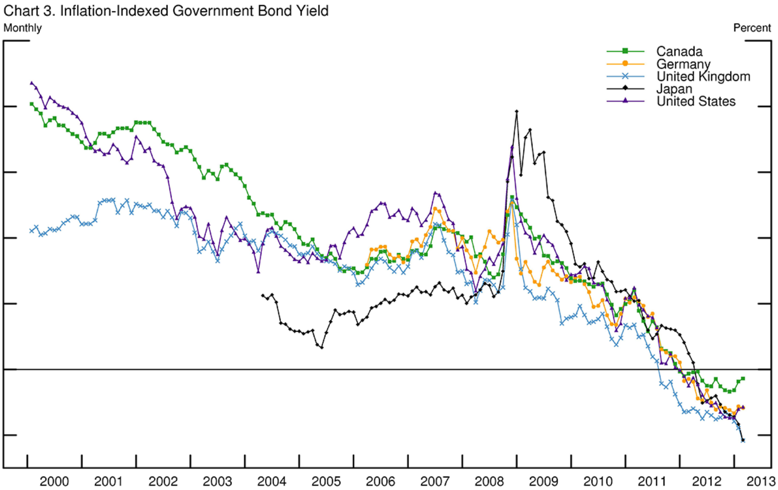
Bond Yields

Long-term interest rates



Source: ECB, Deutsche Bundesbank, Bank of England and Federal Reserve Board.
Latest observation: May 2016.

Inflation is only part of the story ...



Note: Par yields. The maturity for the U.S., U.K., and German bonds is 10 years. The current maturity for the Japanese bond is 5 years, and that for the Canadian bond is 8 years.
Source: Bloomberg.

… and everyone loves German bunds!