ECON 815

Uncertainty and Asset Prices

Winter 2020

Adding Uncertainty

Endowments are now stochastic.

- endowment y_1 in period 1 is known
- two states $s \in \{1, 2\}$ in period 2 with endowment y_s.
- there is a probability distribution (π_1, π_2)

People now maximize expected utility

$$u(c_1) + \beta E[u(c_2)]$$

Key idea:

People face different budget constraints depending on the state s.

Hence, consumption in different states is treated as a different good.

Extending the Framework

With more periods, it is convenient to allow tomorrow's state to depend on today's state.

Example 1: Markov chain with two states

$$\Pi = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}$$

where π_{ij} describes the probability of going from state i today to state j tomorrow.

Example 2: AR(1) process

$$y_t = \rho y_{t-1} + \epsilon_t$$

where $\epsilon_t \sim \mathcal{N}(0, \sigma)$ and $\rho \in (0, 1)$

We then have that tomorrow's expected values are functions of today's state or

$$E_t[y_{t+1}] = E[y_{t+1}|y_t]$$

Decisions under Uncertainty

With two periods, people solve

$$\max E_t[u(c_t) + \beta u(c_{t+1})]$$
$$\text{subject to}$$
$$c_t + a_t = y_t$$
$$c_{s,t+1} = y_{s,t+1} + (1 + r_{t+1})a_t \text{ for all } s$$

where a denote savings in period t and r_{t+1} is a risk-free interest rate.

<u>Solution:</u>

$$-E_t\left[u'(c_t)\right] + E_t\left[\beta u'(c_{t+1})(1 + r_{t+1})\right] = 0$$

or

$$\frac{u'(c_t)}{E_t\left[\beta u'(c_{t+1})\right]} = 1 + r_{t+1}$$

Warning!

1) Arithmetic and harmonic means are different.

$$\frac{u'(c_t)}{E_t\left[\beta u'(c_{t+1})\right]} \neq E_t\left[\frac{u'(c_t)}{\beta u'(c_{t+1})}\right]$$

2) Jensen's Inequality

$$f'' < 0 \quad \Leftrightarrow \quad E[f(x)] < f(E[x])$$

3) Covariance matters

$$E_t[XY] \neq E_t[X]E_t[Y]$$

End Warning!

Asset Prices

What is an asset?

Anything that delivers a payoff in units of consumption across states tomorrow.

Could be a contract, a machine, capital, anything.

More generally, think of a tree.

- price of the tree today is p_t
- payoff consist of dividend tomorrow d_{t+1} ...
- ... and price of the tree tomorrow p_{t+1}

The return from buying a tree is

$$1 + r(s_{t+1}|s_t) = \frac{d(s_{t+1}|s_t) + p(s_{t+1}|s_t)}{p(s_t)}$$

in state s_{t+1} tomorrow given today's state is s_t.

An asset is *risk-free* if it has the same return across all states tomorrow.

Otherwise it is a *risky* asset, with an expected return of

$$E_t[1 + r_{t+1}] = E_t\left[\frac{d_{t+1} + p_{t+1}}{p_t}\right].$$

We use our model – the intertemporal Euler equation, expectations and asset payoffs – to derive a theory of asset prices $\{p(s_{t+1}|s_t)\}$.

Arrow-Debreu Securities

To do so, we first will price elementary securities called *Arrow-Debreu securities*.

- tomorrow's states $s \in \{1, 2, \ldots, S\}$
- today's AD security s pays exactly one unit of consumption in state s tomorrow and nothing in any other state or period
- its price is called the *state price s*
- think of them as one-period, state-contingent zero coupon bonds

All assets can be thought of as portfolios of AD securities.

Key Idea:

If we can price all AD securities (complete markets), we can price any other security through **arbitrage**.

In general, we can rely on the Euler equation to price any payoff which is the consumption-based capital asset pricing model (CCAPM).

Pricing Securities

Suppose there are two states tomorrow and people can only choose AD securities to invest in.

$$\begin{aligned} &\max u(c_t) + \beta E_t[u(c_{t+1})] \\ &\quad \text{subject to} \\ &\qquad c_t + q(1|s_t)a(1|s_t) + q(2|s_t)a(2|s_t) \leq y_t \\ &\qquad c(s_{t+1}|s_t) \leq y_{t+1} + a(s_{t+1}|s_t) \text{ for } s_{t+1} \in \{1,2\} \end{aligned}$$

where $a(s_{t+1}|s_t)$ is the amount of AD security $s_{t+1} \in \{1,2\}$ they buy.

Solution:

$$q(s_{t+1}|s_t) = \frac{\beta\pi(s_{t+1}|s_t)u'(c(s_{t+1}|s_t))}{u'(c_t)}$$

where $\pi(s_{t+1}|s_t)$ is the conditional probability for state s_{t+1} occurring in period t.

Example 1: Consider a one-period risk-free bond that pays 1 unit of consumption in each state tomorrow.

Payoffs for the bond are given by:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Hence, its price is equal to a portfolio consisting of one unit of each of the two AD securities. Thus,

$$q = q(1|s_t) + q(2|s_t) = \frac{\beta\pi(1|s_t)u'(c(1|s_t))}{u'(c_t)} + \frac{\beta\pi(2|s_t)u'(c(2|s_t))}{u'(c_t)}$$

$$= \beta E_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \right].$$

This implies that the risk-free interest rate $q = 1/(1 + r^f)$ rate satisfies

$$1 = E_t \left[\frac{\beta u'(c_{t+1})}{u'(c_t)} \right] (1 + r^f_{t+1})$$

Example 2: Consider any asset with arbitrary payoff across states equal to (x_1, x_2).

Payoffs for this asset are given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Its price must be given by

$$q_x = x_1 q(1|s_t) + x_2 q(2|s_t) = \beta E_t \left[\frac{u'(c_{t+1}) x_{t+1}}{u'(c_t)} \right]$$

Interpret this as equity with payoff x_{t+1}. We then have that the return on equity satisfies

$$1 = \beta E_t \left[\frac{u'(c_{t+1})}{u'(c_t)} (1 + r^e_{t+1}) \right]$$

Consumption Insurance and Risk Premia

We have

$$E[xy] = E[x]E[y] + Cov[xy].$$

This implies for asset pricing that

$$1 = E_t\left[\beta\frac{u'(c_{t+1})}{u'(c_t)}\right]E_t\left[\frac{x_{t+1}}{q_x}\right] + \beta Cov\left[\frac{u'(c_{t+1})}{u'(c_t)}, \frac{x_{t+1}}{q_x}\right]$$

What matters for asset prices?

- average payoff and the covariance of payoffs with consumption
- if covariance is negative, the asset is a hedge against consumption risk which increases the price
- if covariance is positive, people require an additional risk premium to hold the asset which decreases the price

What about CAPM?

We can write the Euler equation as

$$1 = E[m_{t+1}R_{t+1}]$$

where m_{t+1} is often called the stochastic discount factor.

Since $E[m_{t+1}] = 1/(1+r_f)$, we obtain

$$E[R_{t+1}] = \underbrace{(1+r_f)}_{\text{price of time}} + \underbrace{\frac{Cov[m_{t+1}, R_{t+1}]}{Var[m_{t+1}]}}_{\text{quantity of risk}} \left(\underbrace{-\frac{Var[m_{t+1}]}{E[m_{t+1}]}}_{\text{price of risk}} \right)$$

This looks a lot like CAPM!