ECON 815

Two-Period Economies

Winter 2020

Basic Model Set-up

- one good for period 1 (good 1) and period 2 (good 2)
- endowment y_1 and y_2
- people (measure 1) have utility over consumption of these goods

$$u(c_1, c_2) = u(c_1) + \beta u(c_2)$$

- competitive markets for trading these goods ex ante

Budget constraint:

$$p_1 c_1 + p_2 c_2 \leq p_1 y_1 + p_2 y_2$$

Equilibrium

Definition: An equilibrium is a set of prices (p_1, p_2) and an allocation (c_1, c_2) such that

(i) people maximize utility subject to the budget constraint taking prices as given

(ii) markets clear, i.e. $c_1 = y_1$ and $c_2 = y_2$.

Solution:

$$\frac{u'(y_1)}{\beta u'(y_2)} = \frac{p_1}{p_2}$$

Intertemporal Choice

Three markets:

- for goods in period 1
- for goods in period 2
- for saving and borrowing

People have now the budget constraints

$$\begin{aligned} c_1 + s &= y_1 \\ c_2 &= y_2 + (1+r)s \end{aligned}$$

where we have normalized prices within periods $\tilde{p}_1 = \tilde{p}_2 = 1$.

Equilibrium

Definition: An equilibrium is a set of price (interest rate) r and an allocation (c_1, c_2, s) such that

(i) people maximize utility subject to the budget constraints taking the interest rate as given

(ii) markets clear, i.e. $c_1 = y_1$, $c_2 = y_2$ and $s = 0$.

Solution:

$$\frac{u'(y_1)}{\beta u'(y_2)} = (1 + r)$$

This is the central equation of modern macro economics which we will call the **Euler equation**.

Equivalence

Note that

$$\frac{p_1}{p_2} = (1 + r)$$

Hence, both ways of looking at equilibrium are equivalent with interest rates being a ratio of intertemporal prices.

Suppose further that $y_1 = y_2$.

We simply get $(1 + r) = 1/\beta$ or – in terms of the rate of time preference

$$r = \frac{1 - \beta}{\beta} \equiv \theta$$

Sequential vs. Present-Value Budgets

More generally, we have

$$a_{t+1} = (1+r)a_t + y_t - c_t$$

where a_t are total assets owned by a household in period t.

Iterating forward we get

$$a_t = \sum_{i=0}^{\infty} (1+r)^{-i-1} \left(c_{t+i} - y_{t+i}\right)$$

where we assume that $\lim_{i \to \infty} (1+r)^{-i} a_{t+i} = 0$.

Why? This condition rules out that people can borrow or saving unlimited amounts.

Using $t = 0$, we obtain a **present value budget constraint** where current wealth finances future dissavings.

A Closer Look at Consumption

We take interest rates as exogenous where $(1 + r_t) = 1/\beta$.

We assume quadratic utility

$$u(c_t) = ac_t - bc_t^2$$

Euler equation is given by

$$\frac{a - 2bc_t}{\beta(a - 2bc_{t+1})} = (1 + r_t) = \frac{1}{\beta}$$

or $c_t = c_{t+1}$, so that consumption is constant.

Warning: We haven't formally modelled uncertainty, but will introduce it here in an ad-hoc way.

Application I: Certainty Equivalence

If consumption is uncertain tomorrow, we have a random walk

$$E_t[c_{t+1}] = c_t$$

or $c_{t+1} = c_t + \epsilon_{t+1}$ for some ϵ_{t+1} with $E_t[\epsilon_{t+1}] = 0$.

This is related to the concept of "certainty equivalence":

- For the choice only the first moment matters ...
- ... and one can neglect (higher moments of) uncertainty.
- Quadratic utility with linear constraints ...
- ... or linearize the Euler equation.

Application II: Permanent Income Hypothesis

With uncertainty we can use a budget constraint given by

$$a_t = \sum_{i=0}^{\infty} (1+r)^{-i-1} E_t \left[c_{t+i} - y_{t+i} \right]$$

Using the Euler equation, this gives us the **consumption function**

$$c_t = r a_t + r \left(\sum_{i=0}^{\infty} (1+r)^{-i-1} E_t y_{t+i} \right)$$

Consumption depends on the annuity value of current wealth and future expected (permanent) income. Why? With $y_{t+i} = 0$, we get

$$a_{t+1} = (1+r)a_t - c_t = (1+r)a_t - r a_t = a_t$$

Application III: Risk Aversion and Volatility

We now assume CRRA utility

$$u(c_t) = c_t^{1-\alpha}/(1-\alpha)$$

with $\alpha > 0$

From the Euler equation, we have that $c_t^{-\alpha} = \beta(1 + r_{t+1})c_{t+1}^{-\alpha}$ or

$$\alpha \ln\left(\frac{c_{t+1}}{c_t}\right) = \ln\beta + \ln(1 + r_{t+1})$$

Taking first differences, we have that the elasticity of intertemporal substitution is the inverse of the coefficient of relative risk aversion.

For the variance in consumption growth we approximately have

$$V(\ln c_{t+1} - \ln c_t) \simeq \left(\frac{1}{\alpha}\right)^2 V(r_{t+1})$$

If interest rates are much more volatile than consumption, we need α to be fairly large.

Adding Production and Investment

Firms:

- have technology and are owned by people
- borrow goods x from people with endowment y in period 1
- convert these goods into capital $x = k$
- pay back goods plus interest $(1 + r)x$ in period 2

Production:

$$f(k) + (1 - \delta)k$$

Assumptions:

- $f(0) = 0$, $f'(0) = \infty$ and $f'(\infty) = 0$
- $f' > 0$ and $f'' < 0$

Firms maximize profits:

$$\max_k \Pi = \max_k f(k) + k(1-\delta) - (1+r)k$$

Since people get the profits, the budget constraints are

$$\begin{aligned} c_1 &\leq y - x \\ c_2 &\leq (1+r)x + \Pi \end{aligned}$$

Definition: A competitive equilibrium is an interest r and an allocation (x, k, c_1, c_2) such that

(i) people maximize utility taking the interest rate and profits as given

(ii) firms maximize profits taking the interest rate as given

(iii) markets clear, i.e. $c_1 = y - x$, $c_2 = f(k) + (1-\delta)k$ and $x = k$.

Solution

For profit maximization to have a solution that corresponds to an equilibrium, we need to have that

$$f'(k) + (1 - \delta) = (1 + r)$$

or

$$f'(k) = r + \delta$$

<u>Solution:</u>

$$\text{MRT} = f'(k) + (1 - \delta) = (1 + r) = \frac{p_1}{p_2} = \frac{u'(c_1)}{\beta u'(c_2)} = \text{MRS}$$

Since $c_1 = y_1 - k$ and $c_2 = f(k) + (1 - \delta)k$, we have a nonlinear equation in k.

Example

- $u(c) = \ln c$
- $f(k) = k^\alpha$ and $\delta = 1$

Let's look at a **social planner** who simply picks an allocation that maximizes utility for people, while respecting all (technological) constraints.

$$\max_k \ln(y - k) + \beta \ln(k^\alpha)$$

Solution:

$$\begin{aligned} c_1 &= \frac{1}{1+\alpha\beta} y \\ k &= \frac{\alpha\beta}{1+\alpha\beta} y \\ c_2 &= \left(\frac{\alpha\beta}{1+\alpha\beta} y\right)^\alpha \end{aligned}$$

Welfare Theorems

Solution depends on parameters (α, β, y) and the model structure.

Let's use our first-order condition:

$$\text{MRT} = \alpha k^{\alpha-1} = \frac{c_2}{\beta c_1} = \text{MRS}$$

We obtain that the solution of the planner satisfies the Euler equation and the market clearing conditions. Hence, it is an equilibrium with

$$r = \alpha k^{\alpha-1}.$$

This is just a consequence of the two fundamental theorems of welfare economics.

Investment Frictions

1) Adjustment costs

$$k_{t+1} = i_t + (1-\delta)k_t$$
$$f(k_t) = c_t + i_t + \frac{1}{2}\frac{i_t^2}{k_t}$$

2) Financing constraints

$$i_t \leq \gamma k_t$$
or
$$i_t \leq q_t k_t$$

where $\gamma < 1$ and q_t is the (endogenous) market price of capital.

Idea: Both drive a wedge into the Euler equation so that MPK $> r$ and, consequently, there is too little investment/savings.