Economics 442 **Winter 2015**

Topics in Macroeconomics **Thorsten Koeppl**

Answer Key for Assignment 4

Answer to Question 1:

1. From Assignment 2, we know that in a steady state equilibrium overaccumulation of capital occurs if and only if $\frac{1}{2} = \frac{\beta}{1+\beta} > \frac{\alpha}{1-\alpha}$. Hence, this condition requires that $\alpha < \frac{1}{3}$.

   The golden rule capital stock is given by $k_{GR} = \left(\frac{\alpha A}{n}\right)^{\frac{1}{1-\alpha}}$. Hence, we can obtain the optimal PAYG τ by using this expression in the formula for τ

$$\tau(k_{GR}) = (\alpha A)^{\frac{1}{1-\alpha}} n^{\frac{\alpha}{\alpha-1}} \frac{1-2\alpha}{2\alpha}.$$

2. It is straightforward to see that $\tau(k_{GR})$ is increasing in A, but decreasing in n. Note that this result is not obvious. There are two effects at work. First, changes in productivity or the population growth rate change the degree of overaccumulation in the economy, $k_{GR} - k_{SS}$. Second, there is a direct influence of A on income. When A falls, total income falls reducing the tax necessary to change savings. Similarly, there is also a direct influence as n falls, as all else equal an increase in τ is necessary to sustain a transfer a per person of the old generation. This will be come important in the next part of the question.

3. Using the golden rule allocation, pensions are given by

$$a = n\tau(k_{GR}) = (\alpha A)^{\frac{1}{1-\alpha}} n^{\frac{2\alpha-1}{\alpha-1}} \frac{1-2\alpha}{2\alpha} = \alpha^{\frac{1}{1-\alpha}} \frac{1-2\alpha}{2\alpha}$$

   where $\frac{2\alpha-1}{\alpha-1} > 0$, since $\alpha < \frac{1}{3}$. It follows that a is increasing in both A and n.

   Consider a fall in A. This decreases pensions in the optimal PAYG system, as the required tax on savings falls, but n is constant. Consider next a fall in n. Now, there are two effects. A drop in n increases the required tax on savings (an *intensive* margin), but the number of people that are taxed to contribute to the PAYG system falls (an *extensive* margin). For the case considered here, the latter effect dominates and total pensions in the optimal PAYG system need to fall. This gives a road map how to think about necessary changes in payouts as well as contributions to a PAYG when growth in the economy changes due to changes in population and productivity growth.

Answer to Question 2:

1. The first part of the question follows straight from the lecture notes. The optimization problem for any generation t is given by

$$\max_{c_t(t), c_t(t+1), e_t} \ln c_t(t) + \beta \ln c_t(t+1) + \gamma \ln e(t)$$

subject to

$$c_t(t) + s(t+1) + e(t) = w(t)h(t)$$

$$c_t(t+1) = r(t+1)s(t+1)$$

Note that each generation directly likes to spend resources on education of their kids e_t. The FOCs are given by

$$c_t(t) = \frac{1}{1+\beta+\gamma} w(t)h(t)$$

$$c_t(t+1) = \frac{\beta r(t+1)}{1+\beta+\gamma} w(t)h(t)$$

$$e(t) = \frac{\gamma}{1+\beta+\gamma} w(t)h(t)$$

$$s(t+1) = \frac{\beta}{1+\beta+\gamma} w(t)h(t)$$

so that people spend a fraction of their income on savings and split the remaining income between the two "goods", consumption in period t and education for their kids.

Using the production function for human capital, we can write the law of motion for human capital as

$$\begin{aligned} h(t+1) &= Be(t)^\theta h(t)^{1-\theta} \\ &= B\left(\frac{\gamma}{1+\beta+\gamma}\right)^\theta w(t)^\theta h(t) \\ &= gw(t)^\theta h(t) \end{aligned}$$

where g is constant.

2. To derive the law of motion for the capital stock, we need now to take into account that the per-capita capital stock is now defined in terms of efficiency units of labour input $L_t = h_t$ and not with respect to the number of people in the economy which is assumed to be constant here and normalized to 1. Using the first-order condition for savings, this yields

$$k_{t+1} = \frac{s(t+1)}{h(t+1)} = \frac{\beta}{g(1+\beta+\gamma)} w(t)^{1-\theta}$$

Wages are paid according to the marginal product of labour per efficiency unit

$$w(t) = (1-\alpha)Ak_t^\alpha$$

so that the law of motion for the per-capita capital stock is given by

$$k_{t+1} = \frac{\beta}{g(1+\beta+\gamma)} [(1-\alpha)A]^{1-\theta} k_t^{\alpha(1-\theta)}.$$

3. The steady state per-capita capital stock is obtained by setting $k_t = k_{t+1}$. Hence,

$$\bar{k} = \left(\frac{\beta}{g(1+\beta+\gamma)} [(1-\alpha)A]^{1-\theta} \right)^{\frac{1}{1-\alpha(1-\theta)}}.$$

4. The growth rate of human capital is given by

$$\frac{h(t+1)}{h(t)} = gw(t)^\theta.$$

Upon convergence over time to the steady state, the wage rate will be constant and given by $\bar{w} = (1-\alpha)A\bar{k}^\alpha$. Hence, the long-run growth rate of human capital in this economy is given by $g\bar{w}^\theta$.

5. There are now two households with different initial endowment of human capital. Taking the wage rate as given, they maximize their utility as before. Due to log utility, different incomes due to different human capital endowments simply scale savings. Hence, total savings are given by $s_a(t+1) + s_0(t+1)$ and the law of motion for capital is

$$\begin{aligned} k_{t+1} &= \frac{s_a(t+1) + s_0(t+1)}{h_a(t+1) + h_0(t+1)} \\ &= \frac{\frac{\beta}{1+\beta+\gamma} w(t)[h_a(t) + h_0(t)]}{gw(t)^\theta [h_a(t) + h_0(t)]} \\ &= \frac{\beta}{g(1+\beta+\gamma)} w(t)^{1-\theta} \\ &= \frac{\beta}{g(1+\beta+\gamma)} [(1-\alpha)A]^{1-\theta} k_t^{\alpha(1-\theta)}. \end{aligned}$$

Hence, the law of motion for the per-capita capital stock is the same as in the earlier part of the question. As a consequence, the steady state per-capita capital stock and wage rate should also remain the same.

6. The growth rate of human capital is the same for both households and depends only on the wage rate and the constant g. As a consequence, the initial inequality in endowment will be persistent overtime. The long run inequality in human capital is then the same as the inequality in the initial endowment of human capital. In addition, there is persistent inequality in consumption across households since inequality in human capital leads to inequality in income.

7. For the policy to be feasible, the net transfer must equal 0 across the two households. We have then that

$$\bar{w}h_0(0)\tau_0 = \bar{w}h_a(0)\tau_a$$

so that $\tau_0 = 2\tau_a$.

8. Since preferences are identical across the two households, an initial redistribution of income towards equality of income will achieve the same spending and saving behaviour across the two households. Since both households spend the *same* fraction of their income on savings, the steady state capital level – and, hence, the wage rate and total income in the economy – will remain unchanged.

Hence, the policy needs to achieve $h_a(1) = h_0(1)$. This will be the case if and only if the following conditions are fulfilled

$$\begin{aligned} h_a(1) &= Be_a(0)^\theta h_a(0)^{1-\theta} \\ h_0(1) &= Be_0(0)^\theta h_0(0)^{1-\theta} \\ e_a(0) &= \frac{\gamma}{1+\beta+\gamma}\varpi h_a(0)(1-\tau_a) \\ e_0(0) &= \frac{\gamma}{1+\beta+\gamma}\varpi h_0(0)(1+\tau_0) \end{aligned}$$

Thus, $h_a(1) = h_0(1)$ if and only if $h_a(0)(1-\tau_a)^\theta = h_0(0)(1+\tau_0)^\theta$. Together with the condition $\tau_0 = 2\tau_a$, we can then solve the level of taxes which are given by

$$(\tau_a, \tau_0) = \left(\frac{2^{\frac{1}{\theta}} - 1}{2^{\frac{1}{\theta}} + 2}, \frac{2^{\frac{1+\theta}{\theta}} - 2}{2^{\frac{1}{\theta}} + 2} \right).$$

In this economy, where people have identical preferences, any income inequality is purely driven by initial conditions. If people could insure against this inequality before they are born, they would do so. This, however, is not possible. Hence, a transfer system as specified above would achieve a similar outcome as an insurance contract among people before they knew to which group they would belong.

If preferences were different across the groups (for example different γ), any tax as the one above would necessarily alter total savings in the economy, the long-run steady state level of capital, incomes and human capital growth. In general, taxing people with larger investments in education would lead to a lower accumulation of human capital, a lower steady state level of capital and a reduction in income for the economy. This hints to a trade-off between inequality in the economy and overall growth.

Answer to Question 3:

1. We have

$$\frac{k}{y} = \frac{s}{g+\delta}, \; \frac{k}{\tilde{y}} = \frac{s}{g+\delta-\delta s}, \; \tilde{s} = \frac{sg}{g+\delta-\delta s}$$

for the three quantities. For the derivations, see the paper by Krusell and Smith.

2. We have

$$\frac{k}{y} = \frac{\tilde{s}}{g+\tilde{s}\delta}, \; \frac{k}{\tilde{y}} = \frac{\tilde{s}}{g}, \; s = \frac{\tilde{s}(g+\delta)}{g+\tilde{s}\delta}$$

for the three quantities. For the derivations, see again the paper by Krusell and Smith.

3. Let $s = 0.2, \delta = 0.05$. We have for $g = 0.03$

$$\frac{k}{y} = 2.5, \; \frac{k}{\tilde{y}} = 2.857, \; \tilde{s} = 0.0857$$

and for $g = 0.01$

$$\frac{k}{y} = 3.33,\ \frac{k}{\tilde{y}} = 4,\ \tilde{s} = 0.04.$$

Here, the gross savings rate is constant. Hence, as the growth rate falls, the *net* savings rate *decreases*. Indeed, as $g \to 0$, we have that the net savings rate (savings after replacing the depreciated capital stock) goes to 0 as well. The capital/net income ratio increases, but converges to a maximal level.

4. Set $\tilde{s} = 0.2, \delta = 0.05$. We have for $g = 0.03$

$$\frac{k}{y} = 5,\ \frac{k}{\tilde{y}} = 6.67,\ s = 0.4$$

and for $g = 0.01$

$$\frac{k}{y} = 10,\ \frac{k}{\tilde{y}} = 20,\ s = 0.6.$$

This implies as the growth rate falls, the *gross* savings rate *increases*. As $g \to 0$, the gross savings rate approaches 1 which means that all income is used for savings (rather than consumption). The capital/net income ratio explodes.