Economics 442 **Winter 2015**

Topics in Macroeconomics **Thorsten Koeppl**

Answer Key for Assignment 3

Answer to Question 1:

1. The expressions for $\bar{k}$ and k_{GR} can be found in the answer key to Assignment 2. There is overaccumulation whenever $\bar{k} > k_{GR}$ or, equivalently, whenever

$$\frac{\alpha}{1-\alpha} < \frac{\beta}{1+\beta}.$$

This is the case for our parameter values.

2. A PAYG social security system is given by $\tau(t) = a(t)/n$. We first solve the household's decision problem taking as given the PAYG system.

$$\max_{c_t(t), c_t(t+1)} \ln c_t(t) + \beta \ln c_t(t+1)$$

subject to

$$s(t+1) + c_t(t) = w(t) - \tau(t)$$

$$c_t(t+1) = r(t+1)s(t+1) + a(t+1)$$

The solution is given by the FOC and the household's lifetime budget constraint

$$\begin{aligned} \frac{c_t(t+1)}{\beta c_t(t)} &= r(t+1) \\ c_t(t) + \frac{c_t(t+1)}{r(t+1)} &= w(t) - \tau(t) + \frac{a(t+1)}{r(t+1)} \end{aligned}$$

This implies that the household chooses current consumption as a fraction of his lifetime income or

$$c_t(t) = \frac{1}{1+\beta}\left(w(t) - \tau(t) + \frac{a(t+1)}{r(t+1)}\right)$$

Hence, taking interest rates as given, the household anticipates the pension it receives when old and takes into account its net present value when choosing consumption.

The household's saving function is then given by

$$\begin{aligned} s(t+1) &= w(t) - \tau(t) - c_t(t) \\ &= \frac{\beta}{1+\beta}(w(t) - \tau(t)) - \frac{1}{1+\beta}\frac{a(t+1)}{r(t+1)} \end{aligned}$$

where we have used the expression from above. Observe that the decision depends on both, the interest rate $r(t+1)$ and the PAYG system with contributions $\tau(t)$ and future payouts $a(t+1)$.

3. We now need to find a value for τ and a that achieve

$$k_{GR} = \left(\frac{\alpha A}{n}\right)^{\frac{1}{1-\alpha}}.$$

We have from the household's problem that

$$\frac{c_2}{c_1} = rc_1 = w - \tau - sc_2 = rs + a$$

or

$$\frac{rs + a}{w - \tau - s} = r.$$

Consider now a planner that needs to achieve the golden rule. He faces several constraints. First, the last equation gives us a constraint for the planner that expresses that households make optimal decisions. Second, the planner needs to take into account that there is market clearing, or that $s = nk$. Third, wages are given by the marginal product of labour or $w = (1-\alpha)Ak^\alpha$.

Use now k_{GR} in all these constraints and the restriction that the policy needs to satisfy $a = \tau n$. We then have that $r = n$. Using all this information and the constraints in the FOC of the household yields

$$n^2 k_{GR} + \tau n = n(1-\alpha)Ak_{GR}^\alpha - \tau n - n^2 k_{GR}.$$

Rewriting we obtain

$$\tau = \frac{1}{2}(1-\alpha)Ak_{GR}^\alpha - nk_{GR}.$$

It follows immediately that $a = \tau n$. Note that this precisely the expression given in Question 3 of the homework given that we want to achieve k_{GR}.

Remark: The last exercise captures the idea that economic policy tries to implement an efficient policy. Since there is overaccumulation, the policy we found reduces the incentives to save to obtain the golden rule capital stock. Note that policy makers needs to respect how the economy works (i.e. market clearing, optimal decisions by firms and households) and their budget constraint (revenue equals expenditure). These are precisely the constraints we have imposed on the policy maker (aka planner).

Remark: Note that we also obtain the correct consumption allocation. To see this, simply use the expression for τ to obtain consumption when young c_1. This yields

$$c_1 = (1-\alpha)Ak_{GR}^{\alpha} - \tau - nk_{GR} = \frac{1}{2}Ak_{GR}^{\alpha}.$$

Answer to Question 2:

1. For this question, the household *does not* optimize. His savings decision is an exogenous function of earnings $w(t)$. We have that

$$s(t+1) = w(t) - c_t(t) = \frac{\rho}{1+\rho}w(t) = \frac{\rho}{1+\rho}(1-\alpha)Ak(t)^{\alpha}.$$

The law of motion for the per capita capital stock is thus given by

$$k(t+1) = \frac{1}{n}s(t+1) = \left(\frac{(1-\alpha)A}{n}\right)\left(\frac{\rho}{1+\rho}\right)k(t)^{\alpha}.$$

2. The new steady state level is given by

$$\bar{k}_{\rho} = \left(\frac{(1-\alpha)A}{n}\frac{\rho}{1+\rho}\right)^{\frac{1}{1-\alpha}}.$$

It is now lower as $\rho < \beta$. Households – by assumption – consume too much each period and, hence, save too little for when they are old. In that sense they are not rational and behave as if their discount factor is lower, i.e. they value the present more than they should according to their preferences.

The interest rate is now higher as the steady state capital stock is lower and given by

$$\bar{r}_\rho = \alpha A \bar{k}_\rho^{\alpha-1} = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1+\rho}{\rho}\right) n.$$

Remark: In a more elaborate model, this could be expressed as the difference between a short-term (behavioral) discount factor and a long-run (true) discount factor. For further references, consult the literature on Hyperbolic Discounting.

3. We first compare the steady state with the golden rule level of capital. We have

$$\begin{aligned} \bar{k}_\rho &\gtreqless k_{GR} \\ \left(\frac{(1-\alpha)A}{n}\frac{\rho}{1+\rho}\right)^{\frac{1}{1-\alpha}} &\gtreqless \left(\frac{\alpha A}{n}\right)^{\frac{1}{1-\alpha}} \\ \frac{\rho}{1+\rho} &\gtreqless \frac{\alpha}{1-\alpha}. \end{aligned}$$

For the given parameter values, we have that $\bar{k}_\rho = k_{GR}$ so that the economy in fact achieves the golden rule capital stock which is given here by $k_{GR} = 1$.

Remark: Note that this result is by design here. Since $\beta > \rho$, if households behaved optimally, they would save more and the economy would be in a position of overaccumulation. The counteracting force of the bias in excess consumption exactly compensates here the problem of inefficient capital accumulation. Call it a lucky coincidence.

For completeness, we need to check whether the economy achieves an efficient consumption allocation. Total net production is given by

$$\phi(k_{GR}) = (1-\alpha)A = 3$$

with consumption in the golden rule allocation equaling

$$c_{GR}^1 = \frac{1}{1+\beta}\phi(k_{GR}) = \frac{3}{2} = \frac{\beta}{1+\beta}\phi(k_{GR}) = c_{GR}^2.$$

People, however, consume $c_1 = 2$ and $c_2 = 1$ in equilibrium. Hence, while the economy is dynamically efficient, it does not achieve the best consumption allocation. The reason is here that people do not behave optimally for themselves.

Using the savings function given ρ, we need that

$$c_1 = \frac{1}{1+\rho}(w-\tau) = 3/2 \tag{0.1}$$

Since at k_{GR}, we have $w = (1-\alpha)A = 3$, this implies that $\tau = 3/4$.

Private savings are then given by

$$s = w - \tau - c_1 = 3 - 3/4 - 3/2 = 3/4.$$

Hence, the government needs to save an amount of $1/4$ to maintain the capital stock at k_{GR}. Finally, we have to make sure that the old have the right consumption which is given by

$$c_2 = rs + a = 3/4 + a$$

where we have used $r = n = 1$. Hence, the transfer is given by $a = 3/4$.

4. What does this imply for the optimal pension scheme? It is given by $\tau = 3/4$ and $k^s = 1/4$. The pension scheme taxes the young generation and invests the proceeds into capital to pay the generation back $a = 3/4$ when they are old. Hence, the optimal pension scheme is a combination of a fully funded one and a PAYG system. The intuition is that the pension scheme forces people to save more. Given the behavioral specification this lowers people's consumption, but allows the planner to maintain the capital stock at k_{GR}.

   Remark: The last result can be understood from the perspective of too much savings and overaccumulation if the households were to be fully rational. With an exogenous savings function, however, the planner can reduce private savings (by 1/4) and consumption by 1/2 just in the right proportion to achieve the golden rule allocation.

Answer to Question 3:

1. We have with the parameter values that

$$\tau(k) = \frac{\sqrt{k} - 2k}{1 + \sqrt{k}}.$$

The graph below shows the steady state values of k on the x-axis as a function of τ. As discussed, there tend to be two steady states, the lower one $k = 0$ being unstable, the upper one ($k = 1/4$) being stable.

If $k = 0$, there cannot be production in the economy and as a consequence, it is a trivial case. Using the parameter values, the steady state value of $k = 1/4$ is the only SS consistent with the absence of a PAYG system ($\tau = a = 0$).

2. To find the maximum transfer, we differentiate the expression for τ with respect to k and set it equal to 0. This yields the condition

$$\frac{1}{2} - 2\sqrt{k} - k = 0,$$

which is a quadratic equation with the solution given by

$$\sqrt{k} = \sqrt{\frac{3}{2}} - 1.$$

Using this in the expression for τ, we obtain $\tau = \left(1 - \sqrt{\frac{2}{3}}\right)\left(3 - 2\sqrt{\frac{3}{2}}\right)$ as the maximal transfer.

3. For $n = 1/2$, we have now

$$\tau(k) = \frac{\sqrt{k} - k}{1 + 1/2\sqrt{k}}.$$

Using $k = 1/4$, we get $\tau = \frac{1}{5}$.

Suppose we did not increase τ. Then, without a PAYG system, the steady state value of capital would increase. Hence, lower population growth increases the incentives to save and invest. In order to maintain the same capital stock in steady state, one now needs to tax investment which can be done with a PAYG system. When young, people are taxed, but receive a lump-sum transfer when old.

4. Set now $n = 2$, which yields

$$\tau(k) = \frac{\sqrt{k} - 4k}{1 + 2\sqrt{k}}.$$

The maximal capital stock that can now be supported with any PAYG system is given by $k = 1/16$. Hence, we cannot support $k = 1/4$ anymore. The reason here is that the maximum amount that people want to save is given by this value. PAYG systems can *only reduce* savings, but they cannot increase them. The intuition of course is again that pensions "kicking the can down the road" postponing the costs of the initial transfer to the old forever. This leads to lower savings incentives in equilibrium.