Economics 442 **Winter 2015**

Topics in Public Economics **Thorsten Koeppl**

Answer Key for Assignment 2

Answer to Question 1:

1. The household's problem is given by

$$\max_{c_t(t),c_t(t+1),s(t+1)} \ln c_t(t) + \beta \ln c_t(t+1)$$

subject to

$$c_t(t) + s(t+1) \leq w(t)$$
$$c_t(t+1) \leq r(t+1)s(t+1),$$

where $w(t)$ is the wage paid – hence, labor income from supplying one unit of labor – and $r(t+1)$ is the interest rate earned next period from renting out capital acquired through savings $s(t+1)$ when young.

The solution in terms of consumption as a function of prices $(w(t), r(t+1))$ is described by

$$\begin{aligned} \frac{c_t(t+1)}{\beta c_t(t)} &= r(t+1) \\ c_t(t) + \frac{c_t(t+1)}{r(t+1)} &= w(t), \end{aligned}$$

where the first equation is the intertemporal Euler equation and the second one is the intertemporal budget constraint. We have

$$\begin{aligned} c_t(t) &= \frac{1}{1+\beta} w(t) \\ c_t(t+1) &= r(t+1)\frac{\beta}{1+\beta} w(t). \end{aligned}$$

This implies for the savings equation

$$s(t+1) = w(t) - c_t(t) = \frac{\beta}{1+\beta} w(t).$$

Note that for *ln* utility, the household just splits his income into a fixed proportion for current consumption and savings *independent* of the interest rate.

The wage rate is given by the marginal product of labor from the firm's maximization problem or

$$w(t) = \frac{\partial \Pi}{\partial N(t)} = (1-\alpha)Ak(t)^{\alpha}.$$

Hence, capital accumulation is described by

$$\begin{aligned}
K(t+1) &= S(t+1) \\
N_{t+1}k(t+1) &= N_t s(t+1) \\
k(t+1) &= \frac{1}{n}s(t+1) \\
k(t+1) &= \frac{1}{n}\frac{\beta}{1+\beta}w(t) \\
k(t+1) &= \frac{1}{n}\frac{\beta}{1+\beta}(1-\alpha)Ak(t)^{\alpha}.
\end{aligned}$$

Note that second period consumption can be expressed as

$$c_t(t+1) = r(t+1)nk(t+1) = \alpha Ak(t+1)^{\alpha-1}nk(t+1) = n\alpha Ak(t+1)^{\alpha}.$$

2. The steady state level of per capita capital satisfies

$$\bar{k} = \frac{1}{n}\frac{\beta}{1+\beta}(1-\alpha)A\bar{k}^{\alpha}$$

which yields

$$\bar{k} = \left(\frac{(1-\alpha)A}{n}\frac{\beta}{1+\beta}\right)^{\frac{1}{1-\alpha}}.$$

The interest rate is then given by the marginal product of capital from the firm's maximization problem

$$\bar{r} = \alpha A\bar{k}^{\alpha-1} = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1+\beta}{\beta}\right)n.$$

3. The problem for the golden rule allocation is given by

$$\max_{c^1_{GR}, c^2_{GR}, k_{GR}} \ln c^1_{GR} + \beta \ln c^2_{GR}$$

subject to

$$c^1_{GR} + \frac{1}{n}c^2_{GR} = Ak^{\alpha}_{GR} - nk_{GR}.$$

Differentiating, we find that

$$\begin{aligned} \alpha A k_{GR}^{\alpha-1} &= n \\ \frac{c_{GR}^2}{\beta c_{GR}^1} &= n. \end{aligned}$$

Thus,

$$k_{GR} = \left(\frac{\alpha A}{n}\right)^{\frac{1}{1-\alpha}}.$$

Total net production is thus given by

$$\phi(k_{GR}) = A k_{GR}^{\alpha} - n k_{GR} = k_{GR}^{\alpha}\left[A - n k_{GR}^{1-\alpha}\right] = \left(\frac{\alpha A}{n}\right)^{\frac{1}{1-\alpha}} A(1-\alpha).$$

From the feasibility constraint and the Euler equation we then obtain

$$\begin{aligned} c_{GR}^1 &= \frac{1}{1+\beta}\phi(k_{GR}) \\ c_{GR}^2 &= n\frac{\beta}{1+\beta}\phi(k_{GR}). \end{aligned}$$

4. We simply need to compare the steady state level of capita with the golden rule level of capital. We have

$$\begin{aligned} k_{GR} &\gtreqless \bar{k} \\ \left(\frac{\alpha A}{n}\right)^{\frac{1}{1-\alpha}} &\gtreqless \left(\frac{(1-\alpha)A}{n}\frac{\beta}{1+\beta}\right)^{\frac{1}{1-\alpha}} \\ \frac{\alpha}{1-\alpha} &\gtreqless \frac{\beta}{1+\beta}. \end{aligned}$$

If $k_{GR} > (<)\bar{k}$, there is underaccumulation (overaccumulation).

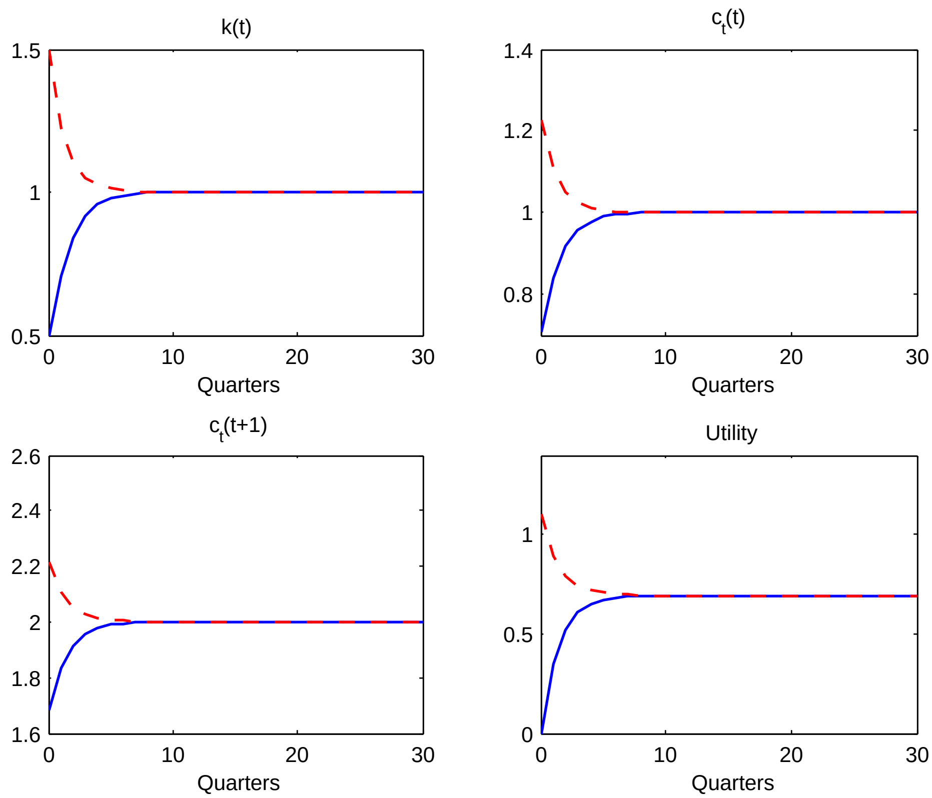
5. Using the parameters, we have the following equations to plot

$$\begin{aligned} k(t+1) &= \sqrt{k(t)} \\ c_t(t) &= \sqrt{k(t)} \\ c_t(t+1) &= 2k(t)^{\frac{1}{4}}. \end{aligned}$$

Finally, the utility for generation $t \geq 0$ is given by

$$u_t = \ln\left(2k(t)^{\frac{3}{4}}\right).$$

For the graphs below, the *blue solid line* corresponds to $k(0) = 0.5$ and the *red dashed line* to $k(0) = 1.5$. "Quarters" are understood to be generations.


k(t)
1.5
1
0.5
0
10
20
30
Quarters
$c_t(t)$
1.4
1.2
1
0.8
0
10
20
30
Quarters
$c_t(t+1)$
2.6
2.4
2.2
2
1.8
1.6
0
10
20
30
Quarters
Utility
1
0.5
0
0
10
20
30
Quarters

Answer to Question 2:

1. The government budget constraint is given by

$$\begin{aligned}\frac{B(t+1)}{1+r(t+1)}+N(t)\tau_1(t) &= B(t)\\ N(t)\left[\frac{b(t+1)}{1+r(t+1)}+\tau_1(t)\right] &= N(t-1)b(t)\\ \frac{b(t+1)}{1+r(t+1)}+\tau_1(t) &= nb(t)\end{aligned}$$

For the steady state, we require that $b(t+1)=b(t)=b$, $r(t+1)=r$ and $\tau_1(t)=\tau_1$. Thus, we have in per capita terms

$$\tau_1 = b\left(\frac{1}{n}-\frac{1}{r}\right).$$

2. Households of any generation t save now in two different instruments, capital $K(t+1)$ and government bonds $B(t+1)$. Hence, aggregate real savings by households are given by

$$\begin{aligned}N(t)s(t+1) &= K(t+1)+\frac{B(t+1)}{r(t+1)}\\ s(t+1) &= nk(t+1)+\frac{b(t+1)}{r(t+1)}\end{aligned}$$

Thus in steady state we obtain

$$\begin{aligned}\tau_1 &= b\left(\frac{1}{n}-\frac{1}{r}\right)\\ s &= n\bar{k}+\frac{b}{r},\end{aligned}$$

where $\bar{k}$ is the steady state level of capital.

3. Recall that the household's total savings are a constant fraction of his life-time income $w(t)-\tau_1(t)$. From the previous parts of the question we obtain

$$s(t+1)=\frac{\beta}{1+\beta}(w(t)-\tau_1(t)).$$

Hence, in steady state, we now have two equations – the government budget constraint and the households savings equation given by

$$\begin{aligned}\frac{\beta}{1+\beta}\left((1-\alpha)A\bar{k}^{\alpha}-\tau_1\right) &= n\bar{k}+\frac{b}{r}\\ b\left(\frac{1}{n}-\frac{1}{r}\right) &= \tau_1.\end{aligned}$$

Furthermore, the interest rate has to be given by

$$r = \alpha A \bar{k}^{\alpha-1}$$

since otherwise there would be an arbitrage opoortunity between real capital and bonds.

Note that this implies that we have two equations in three unknowns, $(\tau_1, b, \bar{k})$. Our system is thus overdetermined. But for any k, the two equations give a unique solution for a government policy, $b(k)$ and $\tau_1(k)$. To achieve the golden rule level of the capital stock, we set $k = k_{GR}$. Note first that

$$r = \alpha A k_{GR}^{\alpha-1} = n.$$

Hence, the transfers to the young τ_1 are exactly zero. This implies that

$$\begin{aligned} b &= n\left(\frac{\beta}{1+\beta}(1-\alpha)Ak_{GR}^{\alpha} - nk_{GR}\right) \\ &= nk_{GR}\left(\frac{\beta}{1+\beta}(1-\alpha)Ak_{GR}^{\alpha-1} - n\right) \\ &= nk_{GR}\left(\frac{\beta}{1+\beta}(1-\alpha)A\frac{n}{\alpha A} - n\right) \\ &= n^2k_{GR}\left(\frac{\beta}{1+\beta}\frac{1-\alpha}{\alpha} - 1\right) \\ &= n^2\left(\frac{\alpha A}{n}\right)^{\frac{1}{1+\alpha}}\left(\frac{\beta}{1+\beta}\frac{1-\alpha}{\alpha} - 1\right). \end{aligned}$$

Suppose first, $b < 0$ or in other words the government saves. This is the case whenever there is underaccumulation. The household is indebted to the government and needs to cover this debt by additional savings in capital. To the contrary, suppose that $b > 0$ or in other words the government is indebted. This is the case whenever there is overaccumulation. The household has wealth in the form of holding government debt and reduces the investment into the capital stock.

Answer to Question 3:

1. Given the parameters specified, the steady state capital level is given by $\bar{k} = \left(\frac{1-\alpha}{2n}\right)^{\frac{1}{1-\alpha}}$. Without the disaster, the consumption for the initial old is $c_{-1}(0) = r(0)s(0) = r(0)nk(0) = n\alpha\bar{k}^{\alpha}$. However, with the disaster, half of the capital stock vanishes. The consumption for the initial old is then only $\tilde{c}_{-1}(0) = n\alpha\left(\frac{\bar{k}}{2}\right)^{\alpha}$.

   Therefore, to keep the initial old's consumption unchanged, the per capita transfer to make up for the consumption loss is given by

$$\tau_2(0) = c_{-1}(0) - \tilde{c}_{-1}(0) = \alpha n\bar{k}^{\alpha}\left(1 - \left(\frac{1}{2}\right)^{\alpha}\right).$$

2. The aggregate tax collected from the young generation needs to be equal to the total transfer to the initial old generation. Hence,

$$\begin{aligned} N_0\tau_1(0) &= N_{-1}\tau_2(0) \\ \tau_1(0) &= \frac{1}{n}\tau_2(0) \\ \tau_1(0) &= \alpha\bar{k}^{\alpha}\left(1 - \left(\frac{1}{2}\right)^{\alpha}\right). \end{aligned}$$

   Note that the per-capita transfer is *independent* of n, the population growth rate.

3. In order for the transfer scheme to be feasible, the per-capita tax cannot exceed the resources of the young generation which are given by the wage $w(0)$. Hence,

$$\tau_1(0) = \alpha\bar{k}^{\alpha}\left(1 - \left(\frac{1}{2}\right)^{\alpha}\right) \leq (1-\alpha)\left(\frac{\bar{k}}{2}\right)^{\alpha} = w(0)$$

   This inequality is fulfilled if and only if

$$\alpha \leq \frac{1}{2^{\alpha}}$$

   Denote $f(\alpha) = 2^{\alpha}\alpha - 1$. The function f is strictly increasing in α, and equals 0 when $\alpha = 0.6412$. Thus, the transfer scheme is feasible when $\alpha \leq 0.6412$. Note that n does not enter the above inequality, so changes in n will not affect the answer.

   The intuition is that α determines the split of output from production among the young who work and the old who own capital. The larger α, the less the young earn. Consequently, for high values of α, the young cannot be taxed enough to make up for the loss suffered by the old on the capital stock.

4. The maximum tax that can be imposed on the old generation is equal to their total resources. Hence, we have

$$\tau_2(0) = \tilde{c}_{-1}(0) = \alpha n \left(\frac{\bar{k}}{2}\right)^{\alpha}.$$

5. Set up the households' optimization problem (similar to 1.a), we can solve the saving as a constant fraction of the total resources for the young generation:

$$\begin{aligned} s_1 &= \frac{\beta}{1+\beta}[w_0 + \frac{\tau_2(0)}{n}] \\ &= \frac{1}{2}\left[(1-\alpha)\left(\frac{\bar{k}}{2}\right)^{\alpha} + \alpha\left(\frac{\bar{k}}{2}\right)^{\alpha}\right] \\ &= \frac{1}{2}\left(\frac{\bar{k}}{2}\right)^{\alpha} \end{aligned}$$

6. Suppose the steady state capital stock is reached in period 1. Then, we need that

$$\bar{k} < k_1 = \frac{s_1}{n} = \frac{1}{2n}\left(\frac{\bar{k}}{2}\right)^{\alpha}.$$

Using the expression for the steady state capital stock $\bar{k}$ from part (a), we obtain

$$2^{\alpha}(1-\alpha) < 1$$

as a necessary and sufficient condition.

Denote $g(a) = 2^{\alpha}(1-\alpha) - 1$. Note that this function is strictly decreasing in $\alpha \in [0,1]$ and reaches its maximum at $\alpha = 0$ where it is 0. So $\bar{k}$ can be achieved in period 1 for any value of $\alpha \in (0,1)$. Hence, it is always possible to go back to the steady state immediately. Since n does not enter the inequality, changes in n will not affect the answer.

<u>Comment 1:</u>

Note that the government does not need to tax the old generation in the full amount in order to achieve the old steady state level of capital in period 1. Indeed, consider a fraction ρ that is taxed. Then, we need that

$$\bar{k} \leq k(1) = \frac{1}{2n}\left[(1-\alpha) + \rho\alpha\right]\left(\frac{\bar{k}}{2}\right)^{\alpha}$$

so that the fraction that needs to be taxed is given by

$$\rho \geq \frac{1}{\alpha}\left(2^{\alpha} - 1\right)(1-\alpha).$$

Note that ρ is decreasing in α. The intuition is again straightforward. For larger values of α, the labour share of income decreases and so does the steady state capital stock.

Comment 2: Consider now that the government invests directly the amount of resources it taxes from the old. We then have that total savings are given by

$$N_0 s + N_{-1}\tau_2(0) = N_1(k_p + k_g) = N_1 k(1)$$

where k_p is the privately held capital stock and k_g is the government held capital stock. Using our expressions from above, we have that

$$k(1) = \frac{1}{2n}(1-\alpha)\left(\frac{\bar{k}}{2}\right)^{\alpha} + \frac{\alpha}{n}\left(\frac{\bar{k}}{2}\right)^{\alpha}.$$

Again, we need $k(1) \geq \bar{k}$. Using our steady state capital stock we obtain as the condition that

$$2^{\alpha}(1-\alpha) \leq (1+\alpha)$$

which is a weaker condition as before. Hence, it has to tax the initial old less than if it gives transfers to the young. The intuition is clear once again. With transfers, the young generation will consume part of the transfer and not invest everything. Note that the government also obtains a free lunch in period 1 from the returns of the capital it has invested.