Economics 442 **Winter 2015**

Topics in Public Economics **Thorsten Koeppl**

Answer Key for Assignment 1

Answer to Question 1:

(a) We do not assume stationary allocations here and derive the general case. Feasibility in period t requires

$$N_t c_t(t) + N_{t-1} c_{t-1}(t) \leq N_t y.$$

Using the fact that $N_t = nN_{t-1}$ we have that

$$c_t(t) + \frac{1}{n} c_{t-1}(t) \leq y.$$

The diagram is the same as shown in the lecture notes.

(b) Consider the stationary allocation $(c_1, c_2) = (\frac{3}{4}y, \frac{1}{2}y)$. The allocation is feasible for $n = 2$, since

$$\frac{3}{4}y + \frac{1}{2}\frac{1}{2}y = y.$$

However, the allocation $(\tilde{c}_1, \tilde{c}_2) = (\frac{1}{4}y, \frac{3}{2}y)$ is also feasible $(\frac{1}{4}y + \frac{3}{2}\frac{1}{2}y = y)$ and delivers strictly more utility to *all* generations (and not only the intial old) than the allocation (c_1, c_2). Hence, the allocation (c_1, c_2) is not Pareto-optimal.

With a stationary allocation, this can also be seen by comparing the intertemporal marginal rate of substitution with the (negative) slope of the feasibility equation, where the later has to be smaller. Here, we have $-\frac{2}{3} > -2$, which shows that (c_1, c_2) is not Pareto-optimal.

(c) We first find the Pareto-optimal allocation that is most preferred by all generations (save the initial old). It solves the problem

$$\max_{c_1, c_2} \sqrt{c_1} + \sqrt{c_2}$$

subject to

$$c_1 + \frac{c_2}{n} = y$$

The FOC for this problem is given by

$$\sqrt{\frac{c_1}{c_2}} = n^2.$$

Using the feasibility condition with $n = 2$, the solution is thus given by

$$\begin{aligned} c_1 &= \frac{1}{3}y \\ c_2 &= \frac{4}{3}y. \end{aligned}$$

This is just the point "A" in our diagram in the lecture. Hence, every other point on the boundary of the feasible set with $c_1 < \frac{1}{3}y$ is also Pareto-optimal. More formally the set of all Pareto-optimal allocations is given by

$$\mathcal{PO} = \left\{ (c_1, c_2) | 0 \leq c_1 < \frac{1}{3}y \wedge c_1 + \frac{c_2}{n} = y \right\}.$$

(d) Since there cannot be any trade, the young and the old simply eat whatever resources they have or

$$\begin{aligned} c_t(t) &\leq y + \tau_t(t) \\ c_{t-1}(t) &\leq \tau_{t-1}(t). \end{aligned}$$

Hence, to achieve the Pareto efficient allocation that we have found in part (d), transfers for the young and the old need to be

$$\begin{aligned} \tau_1 &= c_1 - y = -\frac{2}{3}y \\ \tau_2 &= c_2 = \frac{4}{3}y. \end{aligned}$$

Note that this transfer scheme is feasible, since

$$N_t\tau_1 + N_{t-1}\tau_2 = N_{t-1}y\left(-n\frac{2}{3} + \frac{4}{3}y\right) = 0.$$

In general, given individual endowments, there exists a feasible transfer scheme such that we can achieve any Pareto efficient allocation.

(e) Each generation's maximization problem is now given by

$$\max_{c_t(t), c_{t-1}(t), s_t} \sqrt{c_t(t)} + \sqrt{c_t(t+1)}$$

subject to

$$c_t(t) + s_t \leq y$$
$$c_{t-1}(t) \leq r s_t.$$

Here r is the gross return, so that for $r = 1$, one would just get the initial investment back and for $r > 1$ ($r < 1$) one would make a positive (negative) return.

The solution for this problem is given by the FONC and the life-time budget constraint

$$\begin{aligned} \frac{c_t(t+1)}{c_t(t)} &= r^2 \\ c_t(t) + \frac{1}{r} c_t(t+1) &= y \end{aligned}$$

which yields the stationary allocation $(c_1, c_2) = (1/(1+r)y, r^2/(1+r)y)$.

Comparing the utilities for each generation $t \geq 0$, storage dominates the transfer scheme if and only if

$$\begin{aligned} \sqrt{\frac{1}{1+r} y} + \sqrt{\frac{r^2}{1+r} y} &\geq \sqrt{\frac{1}{1+n} y} + \sqrt{\frac{n^2}{1+n} y} \\ \sqrt{1+r} &\geq \sqrt{1+n} \\ r &\geq n. \end{aligned}$$

(f) If the end of the transfer scheme is publicly announced, it would unravel backwards from period T. The young of generation t have a cost in terms of the transfer $\tau_1 < 0$, but no benefit. Hence, they would prefer putting all their resources into storage. This implies, however, that the young of generation $T-1$ cannot be promised a transfer $\tau_2 > 0$ anymore when they are old. This logic continues until generation 0.

Suppose now that the end of the transfer scheme comes at a complete surprise at T. If the end of the scheme is announced before consumption takes place in period T, the young simply save and we get the solution in part (e). The old of generation $T-1$ would end up with zero consumption in period T. If the end is announced after consumption takes place, the young of generation T bear all the costs.

Remark: As an intermediate case, one could consider that the end is announced before consumption takes place, but after transfers have been made. The young of generation T would then choose their investment into storage optimally, given that their endowment would now be only $y - \tau_1 = 1/3y$. After period T, we would be back to part (e) for the optimal investment behavior.

Answer to Question 2:

(a) With debt financing only, the balanced budget condition in period t is given by

$$\begin{aligned} N(t)\frac{b(t)}{1+r(t)} &= N(t-1)b(t-1) \\ \frac{b(t)}{1+r(t)} &= \frac{b(t-1)}{n}. \end{aligned}$$

(b) The household's problem is given by

$$\begin{aligned} &\max_{c_t(t),c_t(t+1),b(t)} \ln c_t(t) + \ln c_t(t+1) \\ &\text{subject to} \\ &c_t(t) + \frac{b(t)}{1+r(t)} = y_1 \\ &c_t(t+1) = y_2 + b(t) \end{aligned}$$

where the household takes the interest rate $r(t)$ as given.

The FOC is given by

$$\frac{c_t(t+1)}{c_t(t)} = 1 + r(t).$$

(c) The intertemporal budget constraint is given by

$$c_1 + \frac{1}{1+r(t)}c_2 = y_1 + \frac{1}{1+r(t)}y_2.$$

Market clearing yields

$$c_1 + \frac{1}{n}c_2 = y_1 + \frac{1}{n}y_2$$

so that interest rate have to be constant in a stationary perfect foresight equilibrium. Hence, $1 + r(t) = n = 2$ for all t.

The equilibrium consumption allocation is thus given by

$$\begin{aligned} c_1 &= \frac{2n+1}{2n} = \frac{5}{4} \\ c_2 &= \frac{2n+1}{2} = \frac{5}{2}. \end{aligned}$$

The initial debt level consistent with this consumption allocation is

$$b^{SS}(-1) = c_{-1}(0) - y_2 = \frac{2n-1}{2} = \frac{3}{2}.$$

From the government budget constraint, it follows that the sequence of per capita debt levels is thus

$$b^{SS}(t) = b^{SS}(-1) = \frac{3}{2}.$$

The total amount of debt $N(t)b^{SS}(t)$ in the economy, however, increases at rate n.

An important remark. Debt works just like an initial amount of money here – which is really non-interest bearing "debt". One could transfer this economy into a monetary one with a fixed supply of money that's initially given to the old generation. All what is necessary is a vehicle for savings here – whether it is money in constant supply or total debt increasing at the rate of population growth. Note that intertemporal prices (or interest rates) are fixed at n in both cases. This also clarifies that p_{t+1}/p_t is simply a (nominal) interest rate.

(d) To find the lump-sum scheme we just have to look at the budget constraints of the household,

$$\begin{aligned} \tau_1(t) &= y_1 - c_1 = \frac{2n-1}{2n} = \frac{3}{4} \\ \tau_2(t) &= y_2 - c_2 = \frac{1-2n}{2} = -\frac{3}{2}. \end{aligned}$$

It is straightforward to check that these lump-sum transfers satisfy the government budget constraint

$$N(t)\tau_1(t) + N(t-1)\tau_2(t) = 0.$$

(e) I will first describe the algorithm.

We have $b(-1) = 1.01b^{SS}(-1) = 1.515$. The budget constraint of the initial old yields

$$c_{-1}(0) = y_2 + b(-1).$$

For the first iteration, we have $c_{-1}(0) = 2.515$ as the initial condition.

Step 1: Market clearing gives

$$c_t(t) = y_1 + \frac{1}{n}\left(y_2 - c_{t-1}(t)\right).$$

Note that we know $c_{t-1}(t)$. For the first iteration this gives us $c_0(0) = 1.2425$.

Step 2: Solve the household problem given by

$$\begin{aligned} \frac{c_t(t+1)}{c_t(t)} &= 1 + r(t) \\ c_t(t+1) &= y_2 + b(t) = y_2 + b(t-1)\frac{1+r(t)}{n}. \end{aligned}$$

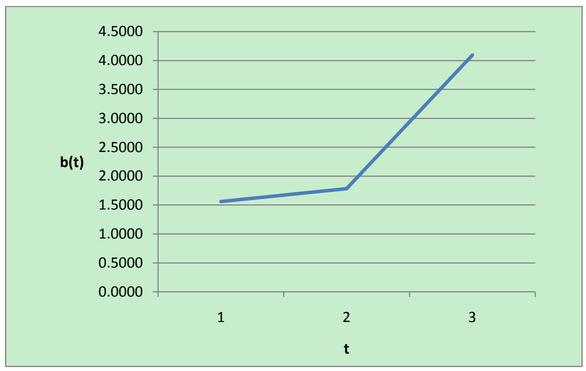
Since we know $c_t(t)$ from Step 1 and $b(t-1)$ from the previous iteration, this can be solved for $c_t(t+1)$ and $1 + r(t)$. For the first iteration, this yields

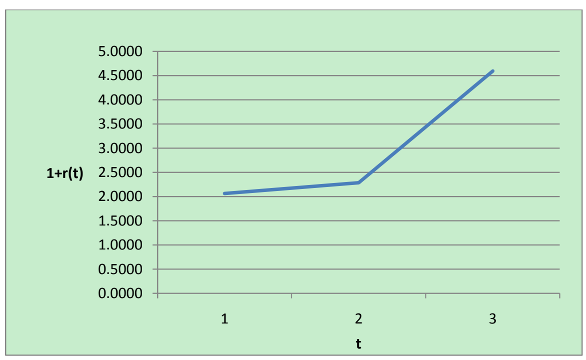
$$\begin{aligned} c_0(1) &= 2.5619 \\ 1 + r(0) &= 2.0619. \end{aligned}$$

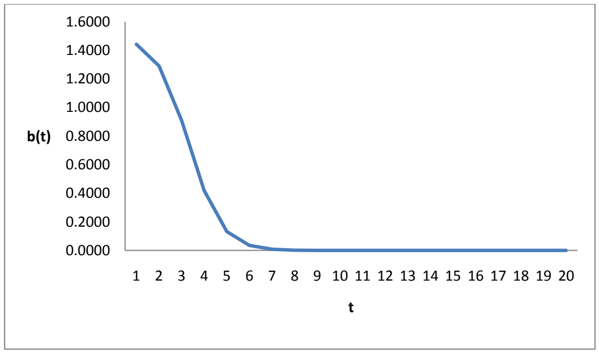
Step 3: Calculate $b(t) = b(t-1)\frac{1+r(t)}{n}$.

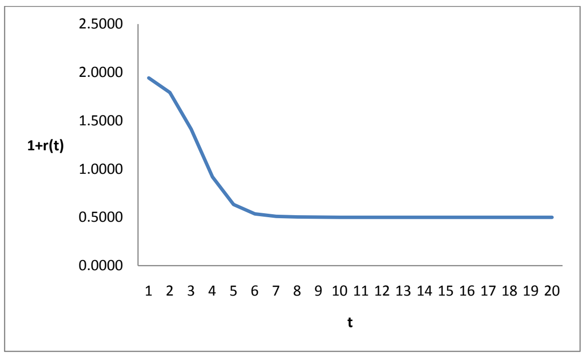
Step 4: Go back to Step 1 with $c_t(t+1)$ and repeat the steps for period $t+1$. In the first iteration, we obtain $b(0) = 1.5619$.

The first set of graphs shows the evolution of debt and interest rates. Debt grows exponentially here, as we start off with a debt level above b^{SS}. In this example, it turns out that after period 2, the debt level is so high that the numbers do not make sense anymore. Debt above the steady state level is not feasible. Note that the allocation gets more and more distorted to the future. Hence, an equilibrium does not exist in any economy with $b(-1) > b^{SS}$.


4.5000
4.0000
3.5000
3.0000
2.5000
b(t)
2.0000
1.5000
1.0000
0.5000
0.0000
1
2
3
t


5.0000
4.5000
4.0000
3.5000
3.0000
1+r(t)
2.5000
2.0000
1.5000
1.0000
0.5000
0.0000
1
2
3
t





(f) The second set of graphs shows the level of debt level and interest rates over time. The economy converges towards autarky, with ever decreasing debt levels and interest rates. In autarky, the debt is zero, but (gross) interest rates are at 0.5. This implies that the policy reduces welfare for all generations. In fact, (net) interest rates need to become negative to keep people from saving (-50%). The last two exercises imply that there is the notion of an optimal debt level for this economy given by b^{SS}.

Answer to Question 3:

(a) Neglecting the initial generation, we can solve the following maximization problem to obtain the stationary optimal allocation:

$$\max_{c_1,c_2} \ln c_1 + \ln c_2$$
$$\text{subject to}$$
$$c_1 + \frac{c_2}{n} = y_1 + \frac{y_2}{n}$$

The first order condition yields

$$\frac{u'(c_1)}{u'(c_2)} = \frac{c_2}{c_1} = n$$

Combining this result with the feasibility constraint, we can solve for the optimal consumption allocation which is given by $c_1^* = \frac{3}{4}, c_2^* = \frac{3}{2}$.

(b) The government budget constraint is given by

$$N_t b_t = N_{t+1} \frac{b_{t+1}}{1+r}$$

Since the per-capita level of debt is fixed at b_0, it must be the case that $\frac{N_{t+1}}{N_t} = n = 2 = 1 + r$. The household problem is now given by

$$\max_{c_1,c_2} \ln c_1 + \ln c_2$$
$$\text{subject to}$$
$$c_1 + \frac{b_0}{1+r} = y_1$$
$$c_2 = y_2 + b_0$$

This implies that the stationary allocation of consumption is again given by the allocation we have found in part (a) with the stationary level of debt being equal to

$$\frac{b_0}{2} = y_1 - c_1 = 1 - \frac{3}{4} = \frac{1}{4}$$

We can then compare the utility level for all generations relative to the autarky case where each generation simply consumes its endowment (with utility level given by 0.

With constant per-capita debt, the utility level for the initial generation is $\ln(1+\frac{b_0}{1+r}) > 0$ since they receive the initial amount that is being raised by debt as a transfer. The utility level for the other generations is $\ln(1-\frac{b_0}{2}) + \ln(1+b_0) = \ln\big((1-\frac{b_0}{2})(1+b_0)\big)$. Therefore, as long as $0 < b_0 < 1$, the debt scheme is welfare improving for everyone.

(c) Let $G_t = \tau_2 N_{t-1}$ be the government spending on the pyramids at time t. We can then write the government budget constraint as follows

$$G_t + B_{t-1} = \frac{B_t}{1+r_t} + \tau_2 N_{t-1}$$

or in per-capita levels

$$g_t + \frac{b_{t-1}}{n} = \frac{b_t}{1+r_t} + \frac{\tau_2}{n}$$

Combining the consumers' budget constraints in both periods,

$$c_t(t) + \frac{b_t}{1+r(t)} = 1$$
$$c_t(t+1) = 1 + b_t - \tau_2$$

we can derive the household's net present value budget constraint to be

$$c_t(t) + \frac{c_t(t+1)}{1+r(t)} = 1 + \frac{1}{1+r(t)} - \frac{\tau_2}{1+r(t)}.$$

(d) The resources to build pyramids are no longer available for private consumption. Hence, we can write the resource constraint as

$$N_t c_t(t) + N_{t-1} c_{t-1}(t) + G_t = N_t y_t(t) + N_{t-1}(y_{t-1}(t)$$

which, under the stationary condition, and using $N_{t-1}\tau_2 = G_t$ is equivalent to

$$c_1 + \frac{c_2}{n} = 1 + \frac{1}{n}(1-\tau_2)$$

Given stationary, we can also rewrite the household's net present value budget constraint as

$$c_1 + \frac{c_2}{1+r} = 1 + \frac{1}{1+r}(1-\tau_2).$$

which again shows that at a stationary equilibrium we need $1 + r = n$.

(e) In the stationary equilibrium, we have again $\frac{u'(c_1)}{u'(c_2)} = \frac{c_2}{c_1} = 1 + r = n = 2$. Substituting this result into the net present value budget constraint that we obtained in part (d) we obtain

$$c_1 = \frac{3}{4} - \frac{1}{4}\tau_2$$
$$c_2 = 2c_1 = \frac{3}{2} - \frac{1}{2}\tau_2$$
$$b_0 = c_2 - (y_2 - \tau_2) = \frac{1}{2}(1 + \tau_2)$$

(f) We can calculate the utility level for the all generations – except the initial generation – to be $\ln(\frac{3}{4} - \frac{1}{4}\tau_2) + \ln(\frac{3}{2} - \frac{1}{2}\tau_2) = \ln\left((\frac{3}{4} - \frac{1}{4}\tau_2)(\frac{3}{2} - \frac{1}{2}\tau_2)\right)$.

As long as this expression is greater than 0, Due to $n = 2$, the first generation is in a majority and will vote for this proposal as long as

$$(\frac{3}{4} - \frac{1}{4}\tau_2)(\frac{3}{2} - \frac{1}{2}\tau_2) \geq 1$$

This gives a quadratic equation, that when solved yields $\tau_2 \leq 3 - 2\sqrt{2}$.

(g) In order to have a more than 2/3 majority vote for the policy, the initial generation must also be voting in favour of this policy. Since the initial proceeds for issuing debt are not paid to the initial old, their consumption level is given by

$$c_{-1} = c_2 = y_2 - \tau_2 < y_2.$$

Hence, for any level of τ_2, they will be worse off than autarky ($c_{-1} = y_2$) and, consequently, will not vote for the proposal.