# ECON 442

# Investment and Growth

Winter 2014

# Production

Aggregate production function:

$$Y(t) = A(t)K(t)^{\alpha}N(t)^{1-\alpha}$$

- $A$ – TFP
- $K$ – capital
- $N$ – labour input

In per-capita terms:

$$y(t) = A(t)k(t)^{\alpha}$$

Capital:

- fully depreciates every period
- needs one period to come online (investment)

Young:

- one unit of labour
- inelastically supplied
- wage $w(t)$
- consume and invest into capital goods

Old:

- cannot work
- sell capital to finance consumption

Three markets:

- Labor market
- Goods market
- Rental market for capital

# Firms

Representative firm runs production to maximize profits.

Firm hires workers and rents capital from the old.

Perfect competition implies that firm takes prices as given:

$$\max_{K(t),N(t)} A(t)K(t)^{\alpha}N(t)^{1-\alpha} - r(t)K(t) - w(t)N(t)$$

Factors are paid their marginal product (zero profits):

$$\begin{aligned} w(t) &= (1-\alpha)A(t)k(t)^{\alpha} \\ r(t) &= \alpha A(t)k(t)^{\alpha-1} \end{aligned}$$

Wages and interest rates are equal to their marginal product.

Young's problem:

$$\max_{c_t(t), c_t(t+1), s(t+1)} u(c_t(t)) + \beta u(c_t(t+1))$$

subject to

$$c_t(t) + s(t+1) = w(t)$$
$$c_t(t+1) = r(t+1)s(t+1)$$

With perfect foresight, household's choice is once again governed by an intertemporal budget constraint:

$$c_t(t) + \frac{c_t(t+1)}{r(t+1)} = w(t)$$

and the intertemporal Euler equation

$$\frac{u'(c_t(t))}{\beta u'(c_t(t+1))} = r(t+1)$$

# Growth

How can we get growth here?

- savings depend on wage earnings
- savings increase tomorrow's capital stock
- wage earnings increase with the total capital stock
- transition to a steady state over time

Dynamic system with capital as a state variable with convergence to a steady state.

Exogenous sources of growth:

- population: $N(t) = nN(t-1)$
- technology: $A(t) = (1+\gamma)A(t-1)$

# Law of Motion for Capital

Savings-function in period $t$:

$$s(t+1) = w(t) - c_t(t)$$

Hence:

$$s(t+1) = s\left(w(t), r(t+1)\right)$$

Capital stock in $t+1$:

$$k(t+1) = \frac{K(t+1)}{N(t+1)} = \frac{N(t)s(t+1)}{N(t+1)} = \frac{1}{n} s\left(w(t), r(t+1)\right)$$

There will be a transition to a steady state where

$$\bar{k} = \frac{1}{n} s\left(w(\bar{k}), r(\bar{k})\right)$$

# Example:

Take log utility.

Savings are a constant fraction of wage earnings.

$$c_t(t) = \frac{1}{1+\beta} w(t)$$

Hence:

$$\begin{aligned} k(t+1) &= \frac{1}{n} s(t+1) \\ &= \frac{1}{n} \frac{\beta}{1+\beta} w(t) \\ &= \frac{1}{n} \frac{\beta}{1+\beta} (1-\alpha) A(t) k(t)^{\alpha} \end{aligned}$$

We obtain a first-order difference equation for capital accumulation over time.

# Steady State

Assume $A(t) = A$ and log utility.

In per capita terms, the steady state level of capital is

$$\bar{k} = \left( \frac{1}{n} \frac{\beta}{1+\beta} (1-\alpha) A \right)^{\frac{1}{1-\alpha}}$$

The overall capital stock is growing in steady state at the fixed growth rate

$$\frac{K(t+1)}{K(t)} = \frac{\bar{k} N(t+1)}{\bar{k} N(t)} = n$$

# Optimality and the Steady State

The steady state satisfies three equations:

$$\begin{aligned}
\frac{u'(c_1)}{\beta u'(c_2)} &= r(\bar{k}) \\
\bar{k} &= \frac{1}{n} s\left(w(\bar{k}), r(\bar{k})\right) \\
A\bar{k}^{\alpha} &= s\left(w(\bar{k}), r(\bar{k})\right) + c_1 + \frac{1}{n} c_2
\end{aligned}$$

Is the SS eql. optimal?

Golden rule allocation:

$$\max_{c_1, c_2, \bar{k}} u(c_1, c_2)$$

subject to

$$A\bar{k}^{\alpha} = n\bar{k} + c_1 + \frac{1}{n} c_2$$

Consider first **productive efficiency**.

$$\max_k \phi(k) = \max_k Ak^\alpha - nk$$

Solution – Golden Rule capital stock

$$\alpha A k_{GR}^{\alpha-1} = n$$

Given $k_{GR}$, **allocative efficiency** requires

$$\begin{aligned} \frac{u'(c_1)}{u'(c_2)} &= n \\ c_1 + \frac{1}{n}c_2 &= \phi(k_{GR}) \end{aligned}$$

ADD GRAPH

# Externality on Savings Decisions

In general, we have that

$$s\,(w(k_{GR}), n) \neq nk_{GR}$$

Individual savings decisions will not lead to socially optimal capital accumulation.

Why?

Generation $t$ does not take into account the impact of its savings decision on future generations.

Two possibilities.

1) Overaccumulation: $\phi'(\bar{k}) < 0$

Solution:

- at some stage $k_t > k_{GR}$, as $k_t \to \bar{k}$
- hence: can increase consumption for all generations after $t$ by reducing capital accumulation
- how? tax savings to lower the return on capital (or lump-sum taxes)

2) Underaccumulation: $\phi'(\bar{k}) > 0$

Problem: Need to increase capital accumulation which implies a decrease in consumption.