ECON 442

Does $r >> g$ matter?

Winter 2015

What does Economic Theory Tell us about r vs. g?

Key idea: savings and investment are endogenous.

Two households: capital owners and workers.

Workers are hand-to-mouth, do not save and simply consume their wages.

Capital owners have preferences given by

$$\sum_{t=0}^{\infty} \beta^t ln(C_t)$$

Production function

$$Y_t = F(K_t, L_t)$$

A Detrended Economy

Assume that all variables grow at a constant rate g.

$$
\begin{aligned}
C_t &= (1+g)^t c_t \\
Y_t &= (1+g)^t y_t \\
K_t &= (1+g)^t k_t
\end{aligned}
$$

Then, the capital owners solve the problem

$$\max_{k_{t+1}} \sum_{t=0}^{\infty} \beta^t ln((1+g)^t c_t)$$

subject to

$$(1+g)k_{t+1} + c_t = k_t + (r_t - \tau)k_t$$

where τ is a tax on capital income.

Workers just consume their income without saving so that

$$w_t = y_t - r_t k_t + \tau k_t.$$

Long-run relationsship between r and g

Taking a FOC condition yields

$$\frac{(1+g)}{\beta}\frac{c_{t+1}}{c_t} = (1 + r_t - \tau)$$

In steady state, we have

$$\frac{(1+g)}{\beta} = (1 + r - \tau)$$

where ρ is the rate of time preference.

Conclusion:

Interest rates are endogenous and given by

$$r \approx \rho + g + \tau$$

with consumption being given by

$$c \approx (r - \tau - g)k$$

Taxing Capital a Bad Idea?

Suppose we increase τ.

Then, r increases 1-1, so that $r - \tau$ is constant.

However, $r = f'(k)$ and capital will fall.

Hence, both c and w fall.

But, w/c is increasing in τ for a concave production function.

Conclusion:

Taxing capital might decrease "inequality", but it will also decrease output.

Proof for $\partial w/\partial \tau < 0$

Note that $(r - \tau)$ is constant when τ changes and $\partial k/\partial \tau < 0$.

Also note that

$$\frac{\partial y}{\partial \tau} = \alpha A k^{\alpha-1} l^{1-\alpha} \frac{\partial k}{\partial \tau} = \alpha \frac{y}{k} \frac{\partial k}{\partial \tau}$$

Hence:

$$\begin{aligned} \frac{\partial w}{\partial \tau} &= \frac{\partial y}{\partial \tau} - (r - \tau) \frac{\partial k}{\partial \tau} \\ &= \alpha \frac{y}{k} \frac{\partial k}{\partial \tau} - (r - \tau) \frac{\partial k}{\partial \tau} \\ &= \left(\alpha \frac{y}{k} - (r - \tau) \right) \frac{\partial k}{\partial \tau} \\ &< 0 \end{aligned}$$

since $\alpha y/k = r$.

Proof for $\partial(w/c)/\partial\tau > 0$

Relative consumption is defined by

$$\frac{w}{c} = \frac{y - rk + \tau k}{(r - \tau - g)k}$$

The change is proportional to

$$\begin{aligned}
&(r-\tau-g)k\left(\frac{\partial y}{\partial \tau} - (r-\tau)\frac{\partial k}{\partial \tau}\right) - (y - rk + \tau k)(r-\tau-g)\frac{\partial k}{\partial \tau} = \\
&= k\frac{\partial y}{\partial \tau} - y\frac{\partial k}{\partial \tau} \\
&= (\alpha - 1)y\frac{\partial k}{\partial \tau} \\
&> 0
\end{aligned}$$

since $\alpha < 1$.

Inequality and Pareto Distributions

Suppose we have

$$\mathcal{P}[\text{Age} > x] = e^{-dx}$$

and for income

$$y = e^{\mu x}.$$

Inequality of income is given by a Pareto distribution.

$$\begin{aligned}\mathcal{P}[\text{Income} > y] &= \mathcal{P}[\text{Age} > x(y)] = \mathcal{P}[\text{Age} > \frac{1}{\mu}\ln y] \\ &= e^{-\frac{d}{\mu}\ln y} = y^{-\frac{d}{\mu}} \\ &= y^{-\frac{1}{\eta}}\end{aligned}$$

Pareto coefficient $\eta < 1$ measures inequality.

Example: top a% of b% get a fraction of income for all of b equal to

$$S(a,b) = \left(\frac{a}{b}\right)^{\eta - 1}$$

A Theory of Wealth Inequality

We assume that

- technology grows at rate g
- population grows at rate n
- people die at rate d

Suppose wealth evolves according to

$$a_t(x) = a_{t-x}(0)e^{(r-\tau-\alpha)x}$$

where x is age, $a_{t-x}(0)$ is inherited wealth and a fraction α of assets is consumed.

Inherited wealth is given by

$$a_{t-x}(0) = \bar{a}k_{t-x} = \bar{a}k_t e^{-gx}$$

The wealth distribution at time t is given by

$$\begin{aligned}\mathcal{P}[\text{Wealth} > a] &= \mathcal{P}[\text{Age} > x(a)] \\ &= e^{-(n+d)x(a_t)} \\ &= \left(\frac{a}{\bar{a}k_t}\right)^{-\frac{n+d}{r-g-\tau-\alpha}}\end{aligned}$$

where $n + d$ is the birth rate in the economy.

Pareto coefficient is given by

$$\eta = \frac{r - g - \tau - \alpha}{n + d}$$

Conclusion:

The Pareto coefficient (i.e. inequality) rises with $r - g$ and falls with $n + d$.