Economics 421 **Fall 2015**

Topics in Macroeconomics **Thorsten Koeppl**

Answer Key for Assignment 4

Answer to Question 1:

1. The social planner maximizes the utility of a representative generation:

$$\max_{c_1,c_2,s} \ln c_1 + \beta \ln c_2$$

$$\text{subject to}$$

$$C_1^{\omega} N + rsN = c_1 N + c_2 N + sN$$

$$C_1 = Nc_1$$

where N is the population size.

The first-order condition is given by

$$\frac{c_2}{\beta c_1} = 1 - \omega c_1^{\omega-1} N^{\omega}.$$

Using the fact that $C_1 = Nc_1$, $r = 1$ and the resource constraint, we obtain

$$(Nc_1)^{\omega} + s = c_1 + (\beta c_1 - \beta\omega c_1^{\omega} N^{\omega}) + s$$

$$c_1^{\omega} N^{\omega}(1 + \beta\omega) = (1 + \beta)c_1$$

$$c_1 = \left(\frac{N^{\omega}(1+\beta\omega)}{1+\beta}\right)^{\frac{1}{1-\omega}}.$$

Finally, solving for c_2 as a function of c_1

$$c_2 = \beta(1 - \omega c_1^{\omega-1} N^{\omega})c_1$$

$$c_2 = \beta\left(1 - \omega\left(\frac{N^{\omega}(1+\beta\omega)}{1+\beta}\right)^{\frac{\omega-1}{1-\omega}} N^{\omega}\right)c_1$$

$$c_2 = \beta\left(1 - \omega\left(\frac{1+\beta}{N^{\omega}(1+\beta\omega)}\right)N^{\omega}\right)c_1$$

$$c_2 = \beta\left(1 - \left(\frac{\omega+\beta\omega}{1+\beta\omega}\right)\right)c_1$$

$$c_2 = \beta\left(\frac{1+\beta\omega-\omega-\beta\omega}{1+\beta\omega}\right)c_1$$

$$c_2 = \beta\left(\frac{1-\omega}{1+\beta\omega}\right)c_1.$$

Setting $N = 2$, the optimal stationary allocation is hence characterized by

$$c_1 = \left(\frac{2^{\omega}(1+\beta\omega)}{1+\beta} \right)^{\frac{1}{1-\omega}}$$
$$c_2 = \beta \left(\frac{1-\omega}{1+\beta\omega} \right) c_1.$$

2. The young generation maximizes utility

$$\max_{c_1, c_2, s} \ln c_1 + \beta \ln c_2$$
$$\text{subject to}$$
$$c_1 + s = w$$
$$c_2 = rs$$

which yields the first-order condition

$$\frac{c_2}{\beta c_1} = 1.$$

Substituting the FOC into the budget constraint, we can solve for individual demand

$$c_1 = \frac{1}{1+\beta} w$$
$$c_2 = \frac{\beta}{1+\beta} w.$$

Combining the wage from the firm's problem, i.e. $w = C_1^{\omega}$, with the market clearing condition $C_1 = c_1 N$ yields

$$w = c_1^{\omega} N^{\omega}$$

so that

$$c_1 = \left(\frac{N^{\omega}}{1+\beta} \right)^{\frac{1}{1-\omega}}$$
$$c_2 = \beta \left(\frac{N^{\omega}}{1+\beta} \right)^{\frac{1}{1-\omega}}.$$

Finally, using $N = 2$,

$$c_1 = \left(\frac{2^{\omega}}{1+\beta} \right)^{\frac{1}{1-\omega}}$$
$$c_2 = \beta \left(\frac{2^{\omega}}{1+\beta} \right)^{\frac{1}{1-\omega}}.$$

3. The social planner's allocation is given by

$$c_1^{SP} = 1.125$$
$$c_2^{SP} = 0.375$$

while the competitive equilibrium allocation yields

$$c_1^{CE} = 0.5$$
$$c_2^{CE} = 0.5.$$

As expected, aggregate demand is inefficiently low in the first period in the competitive equilibrium. In general, implied savings are either too high or too low depending on the income effect that arises from higher output due to the demand externality. The income effect depends on the value of ω, and so savings are too high (low) as $\omega \to 0$ ($\omega \to 1$). In this case, with $\omega = 1/2$, implied savings are too high as $c_2^{SP} < c_2^{CE}$.

4. With the tax τ and the transfer T, the young generation now maximizes:

$$\max_{c_1, c_2, s} \ln c_1 + \beta \ln c_2$$

subject to

$$c_1 + s = w + T$$
$$c_2 = r(1-\tau)s$$

which yields the FOC

$$\frac{c_2}{\beta c_1} = 1 - \tau. \tag{0.1}$$

Now the optimal tax is set in such a way as to equate the competitive equilibrium's FOC and the social planner's FOC or

$$\frac{c_2}{\beta c_1} = 1 - \tau = 1 - \omega c_1^{\omega-1} N^{\omega}$$

so that

$$\tau = \omega c_1^{\omega-1} N^{\omega}.$$

The consumption by the young, c_1, must equal the one derived in part (a)

$$\tau = \omega \left(\left(\frac{N^{\omega}(1+\beta\omega)}{1+\beta} \right)^{\frac{1}{1-\omega}} \right)^{\omega-1} N^{\omega}$$
$$\tau = \omega \left(\frac{N^{\omega}(1+\beta\omega)}{1+\beta} \right)^{\frac{\omega-1}{1-\omega}} N^{\omega}$$
$$\tau = \omega \left(\frac{N^{\omega}(1+\beta\omega)}{1+\beta} \right)^{-1} N^{\omega}$$
$$\tau = \omega \frac{1+\beta}{N^{\omega}(1+\beta\omega)} N^{\omega}$$
$$\tau = \frac{\omega(1+\beta)}{1+\beta\omega}.$$

It is instructive to check that with tax τ the competitive equilibrium allocation corresponds to the allocation of the social planner. Note that the first-order condition of the young generation given tax τ corresponds to the planner's first order condition. What remains to be verified is that the budget constraint of the young generation given τ is identical to the resource constraint that the planner faces. Then the solution to the two problems must be identical. Using $r = 1$, the equilibrium wage rate and the fact that $NT = N\tau s$, the budget constraint becomes

$$c_1 = w + T - s = w + (\tau - 1)s = w - c_2 = C_1^{\omega} - c_2 = (2c_1)^{\omega} - c_2$$

which is identical to the resource constraint the planner faces.

5. The tax τ is strictly positive regardless of the value of β

$$\tau = \frac{\omega(1+\beta)}{1+\beta\omega} > 0.$$

Differentiating this expression with respect to β we obtain

$$\frac{\partial \tau}{\partial \beta} = \frac{\omega(1+\beta\omega) - \omega^2(1+\beta)}{(1+\beta\omega)^2} = \frac{\omega(1-\omega)}{(1+\beta\omega)^2} > 0$$

as long as $\omega < 1$.

Hence, when people have a higher preference to save, the tax necessary to achieve the optimal allocation needs to increase. This is intuitive. As β increases, people prefer to save more. They do not take into account, however, that their savings decision has a negative externality on current output. Hence, current output decreases if people save more as the externality gets larger. Consequently, one has to levy higher taxes to induce people to internalize the externality.

6. The tax τ is strictly positive regardless of the value of ω

$$\tau = \frac{\omega(1+\beta)}{1+\beta\omega} > 0.$$

Differentiating this expression with respect to ω we obtain

$$\frac{\partial \tau}{\partial \omega} = \frac{(1+\beta)(1+\beta\omega) - \omega(1+\beta)(\beta)}{(1+\beta\omega)^2} = \frac{1+\beta}{(1+\beta\omega)^2} > 0.$$

Hence, when the tax must decrease with a fall in ω. The reason is again striaghtforward. The parameter ω expresses how strong the demand externality is. As $\omega \to 0$, the externality in fact disappears altogether. As ω falls savings matter less for reducing current outcome and lower taxes are required to correct the negative externality that arises from savings.

Answer to Question 2:

1. The first part of the question follows straight from the lecture notes. The optimization problem for any generation t is given by

$$\max_{c_t(t), c_t(t+1), e_t} \ln c_t(t) + \beta \ln c_t(t+1) + \gamma \ln e(t)$$

subject to

$$c_t(t) + s(t+1) + e(t) = w(t)h(t)$$
$$c_t(t+1) = r(t+1)s(t+1)$$

Note that each generation directly likes to spend resources on education of their kids e_t. The FOCs are given by

$$c_t(t) = \frac{1}{1+\beta+\gamma} w(t)h(t)$$
$$c_t(t+1) = \frac{\beta r(t+1)}{1+\beta+\gamma} w(t)h(t)$$
$$e(t) = \frac{\gamma}{1+\beta+\gamma} w(t)h(t)$$
$$s(t+1) = \frac{\beta}{1+\beta+\gamma} w(t)h(t)$$

so that people spend a fraction of their income on savings and split the remaining income between the two "goods", consumption in period t and education for their kids.

Using the production function for human capital, we can write the law of motion for human capital as

$$\begin{aligned} h(t+1) &= Be(t)^\theta h(t)^{1-\theta} \\ &= B\left(\frac{\gamma}{1+\beta+\gamma}\right)^\theta w(t)^\theta h(t) \\ &= gw(t)^\theta h(t) \end{aligned}$$

where g is constant.

2. To derive the law of motion for the capital stock, we need now to take into account that the per-capita capital stock is now defined in terms of efficiency units of labour input $L_t = h_t$ and not with respect to the number of people in the economy which is assumed to be constant here and normalized to 1. Using the first-order condition for savings, this yields

$$k_{t+1} = \frac{s(t+1)}{h(t+1)} = \frac{\beta}{g(1+\beta+\gamma)} w(t)^{1-\theta}$$

Wages are paid according to the marginal product of labour per efficiency unit

$$w(t) = (1-\alpha)Ak_t^\alpha$$

so that the law of motion for the per-capita capital stock is given by

$$k_{t+1} = \frac{\beta}{g(1+\beta+\gamma)} [(1-\alpha)A]^{1-\theta} k_t^{\alpha(1-\theta)}.$$

3. The steady state per-capita capital stock is obtained by setting $k_t = k_{t+1}$. Hence,

$$\bar{k} = \left(\frac{\beta}{g(1+\beta+\gamma)} [(1-\alpha)A]^{1-\theta} \right)^{\frac{1}{1-\alpha(1-\theta)}}.$$

4. The growth rate of human capital is given by

$$\frac{h(t+1)}{h(t)} = gw(t)^{\theta}.$$

Upon convergence over time to the steady state, the wage rate will be constant and given by $\bar{w} = (1-\alpha)A\bar{k}^{\alpha}$. Hence, the long-run growth rate of human capital in this economy is given by $g\bar{w}^{\theta}$.

5. There are now two households with different initial endowment of human capital. Taking the wage rate as given, they maximize their utility as before. Due to log utility, different incomes due to different human capital endowments simply scale savings. Hence, total savings are given by $s_a(t+1) + s_0(t+1)$ and the law of motion for capital is

$$\begin{aligned} k_{t+1} &= \frac{s_a(t+1) + s_b(t+1)}{h_a(t+1) + h_b(t+1)} \\ &= \frac{\frac{\beta}{1+\beta+\gamma} w(t)[h_a(t) + h_b(t)]}{gw(t)^{\theta}[h_a(t) + h_b(t)]} \\ &= \frac{\beta}{g(1+\beta+\gamma)} w(t)^{1-\theta} \\ &= \frac{\beta}{g(1+\beta+\gamma)} [(1-\alpha)A]^{1-\theta} k_t^{\alpha(1-\theta)}. \end{aligned}$$

Hence, the law of motion for the per-capita capital stock is the same as in the earlier part of the question. As a consequence, the steady state per-capita capital stock and wage rate should also remain the same.

6. The growth rate of human capital is the same for both households and depends only on the wage rate and the constant g. As a consequence, the initial inequality in endowment will be persistent overtime. The long run inequality in human capital is then the same as the inequality in the initial endowment of human capital. In addition, there is persistent inequality in consumption across households since inequality in human capital leads to inequality in income.

7. For the policy to be feasible, the net transfer must equal 0 across the two households. We have then that

$$\bar{w}h_b(0)\tau_b = \bar{w}h_a(0)\tau_a$$

so that $\tau_b = 2\tau_a$.

8. Since preferences are identical across the two households, an initial redistribution of income towards equality of income will achieve the same spending and saving behaviour across the two households. Since both households spend the *same* fraction of their income on savings, the steady state capital level – and, hence, the wage rate and total income in the economy – will remain unchanged.

Hence, the policy needs to achieve $h_a(1) = h_b(1)$. This will be the case if and only if the following conditions are fulfilled

$$\begin{aligned} h_a(1) &= Be_a(0)^\theta h_a(0)^{1-\theta} \\ h_0(1) &= Be_0(0)^\theta h_b(0)^{1-\theta} \\ e_a(0) &= \frac{\gamma}{1+\beta+\gamma}\bar{w}h_a(0)(1-\tau_a) \\ e_0(0) &= \frac{\gamma}{1+\beta+\gamma}\bar{w}h_b(0)(1+\tau_b) \end{aligned}$$

Thus, $h_a(1) = h_b(1)$ if and only if $h_a(0)(1-\tau_a)^\theta = h_b(0)(1+\tau_b)^\theta$. Together with the condition $\tau_b = 2\tau_a$, we can then solve the level of taxes which are given by

$$(\tau_a, \tau_b) = \left(\frac{2^{\frac{1}{\theta}} - 1}{2^{\frac{1}{\theta}} + 2}, \frac{2^{\frac{1+\theta}{\theta}} - 2}{2^{\frac{1}{\theta}} + 2} \right).$$

In this economy, where people have identical preferences, any income inequality is purely driven by initial conditions. If people could insure against this inequality before they are born, they would do so. This, however, is not possible. Hence, a transfer system as specified above would achieve a similar outcome as an insurance contract among people before they knew to which group they would belong.

9. If preferences were different across the groups (for example different γ), any tax as the one above would necessarily alter total savings in the economy, the long-run steady state level of capital, incomes and human capital growth. In general, taxing people with larger investments in education would lead to a lower accumulation of human capital, a lower steady state level of capital and a reduction in income for the economy. This hints to a trade-off between inequality in the economy and overall growth.