Economics 421 **Fall 2015**

Topics in Macroeconomics **Thorsten Koeppl**

Answer Key for Assignment 3

Answer to Question 1:

1. The household's problem is given by

$$\max_{c_t(t),c_t(t+1),s(t+1)} \ln c_t(t) + \beta \ln c_t(t+1)$$
subject to
$$c_t(t) + s(t+1) \leq w(t)$$
$$c_t(t+1) \leq r(t+1)s(t+1),$$

where $w(t)$ is the wage paid – hence, labor income from supplying one unit of labor – and $r(t+1)$ is the interest rate earned next period from renting out capital acquired through savings $s(t+1)$ when young.

The solution in terms of consumption as a function of prices $(w(t), r(t+1))$ is described by

$$\begin{aligned} \frac{c_t(t+1)}{\beta c_t(t)} &= r(t+1) \\ c_t(t) + \frac{c_t(t+1)}{r(t+1)} &= w(t), \end{aligned}$$

where the first equation is the intertemporal Euler equation and the second one is the intertemporal budget constraint. We have

$$\begin{aligned} c_t(t) &= \frac{1}{1+\beta} w(t) \\ c_t(t+1) &= r(t+1)\frac{\beta}{1+\beta} w(t). \end{aligned}$$

This implies for the savings equation

$$s(t+1) = w(t) - c_t(t) = \frac{\beta}{1+\beta} w(t).$$

Note that for log utility, the household just splits his income into a fixed proportion for current consumption and savings *independent* of the interest rate.

The wage rate is given by the marginal product of labor from the firm's maximization problem or

$$w(t) = (1-\alpha)Ak(t)^{\alpha}.$$

Hence, capital accumulation is described by

$$\begin{aligned}
N_{t+1}k(t+1) &= N_t s(t+1) \\
k(t+1) &= \frac{1}{n}s(t+1) \\
k(t+1) &= \frac{1}{n}\frac{\beta}{1+\beta}w(t) \\
k(t+1) &= \frac{1}{n}\frac{\beta}{1+\beta}(1-\alpha)Ak(t)^{\alpha}.
\end{aligned}$$

Note that second period consumption can be expressed as

$$c_t(t+1) = r(t+1)nk(t+1) = \alpha Ak(t+1)^{\alpha-1}nk(t+1) = n\alpha Ak(t+1)^{\alpha}.$$

2. The steady state level of per capita capital satisfies

$$\bar{k} = \frac{1}{n}\frac{\beta}{1+\beta}(1-\alpha)A\bar{k}^{\alpha}$$

which yields

$$\bar{k} = \left(\frac{1}{n}\frac{\beta}{1+\beta}(1-\alpha)A\right)^{\frac{1}{1-\alpha}}.$$

The interest rate is then given by the marginal product of capital from the firm's maximization problem

$$\bar{r} = \alpha A\bar{k}^{\alpha-1} = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1+\beta}{\beta}\right)n.$$

3. The problem for the golden rule allocation is given by

$$\max_{c^1_{GR},c^2_{GR},k_{GR}} \quad \ln c^1_{GR} + \beta \ln c^2_{GR}$$

subject to

$$c^1_{GR} + \frac{1}{n}c^2_{GR} = Ak^{\alpha}_{GR} - nk_{GR}.$$

Differentiating, we find that

$$\begin{aligned}\alpha A k_{GR}^{\alpha-1} &= n \\ \frac{c_{GR}^2}{\beta c_{GR}^1} &= n.\end{aligned}$$

Thus,

$$k_{GR} = \left(\frac{\alpha A}{n}\right)^{\frac{1}{1-\alpha}}.$$

Total net production is thus given by

$$\phi(k_{GR}) = Ak_{GR}^{\alpha} - nk_{GR} = k_{GR}^{\alpha}\left[A - nk_{GR}^{1-\alpha}\right] = (1-\alpha)A\left(\frac{\alpha A}{n}\right)^{\frac{1}{1-\alpha}}.$$

From the feasibility constraint and the Euler equation we then obtain

$$\begin{aligned}c_{GR}^1 &= \frac{1}{1+\beta}\phi(k_{GR}) \\ c_{GR}^2 &= n\frac{\beta}{1+\beta}\phi(k_{GR}).\end{aligned}$$

4. We simply need to compare the steady state level of capita with the golden rule level of capital. We have

$$\begin{aligned}k_{GR} &\gtreqless \bar{k} \\ \left(\frac{\alpha A}{n}\right)^{\frac{1}{1-\alpha}} &\gtreqless \left(\frac{(1-\alpha)A}{n}\frac{\beta}{1+\beta}\right)^{\frac{1}{1-\alpha}} \\ \frac{\alpha}{1-\alpha} &\gtreqless \frac{\beta}{1+\beta}.\end{aligned}$$

If $k_{GR} > (<)\bar{k}$, there is underaccumulation (overaccumulation).

Answer to Question 2:

1. The government budget constraint is given by

$$\begin{aligned}\frac{B(t+1)}{r(t+1)} + N_t\tau_1(t) &= B(t) \\ N_t\left[\frac{b(t+1)}{r(t+1)} + \tau_1(t)\right] &= N_{t-1}b(t) \\ \frac{b}{r(t+1)} + \tau_1(t) &= \frac{b}{n}\end{aligned}$$

where we have used the fact that $b(t-1) = b(t) = b$. Thus we have in per capita terms

$$\tau_1(t) = b\left(\frac{1}{n} - \frac{1}{r(t+1)}\right).$$

For the steady state this yields

$$\tau_1 = b\left(\frac{1}{n} - \frac{1}{r}\right)$$

2. Households of any generation t save now in two different instruments, capital $K(t+1)$ and government bonds $B(t+1)$. Hence, aggregate real savings by households have to equal total investments and are given by

$$\begin{aligned}N_t s(t+1) &= N_{t+1}k(t+1) + \frac{B(t+1)}{r(t+1)} \\ s(t+1) &= nk(t+1) + \frac{b(t+1)}{r(t+1)}\end{aligned}$$

Thus in steady state we obtain

$$s = n\bar{k} + \frac{b}{r}.$$

3. From Question 1, we know that the household's total savings are a constant fraction of his income which is now given by $w(t) - \tau_1(t)$. Hence,

$$s(t+1) = \frac{\beta}{1+\beta}(w(t) - \tau_1(t)).$$

Hence, in steady state, we obtain that

$$\begin{aligned}&\frac{\beta}{1+\beta}\left((1-\alpha)A\bar{k}^{\alpha} - \tau_1\right) = n\bar{k} + \frac{b}{r} \\ &\tau_1 = b\left(\frac{1}{n} - \frac{1}{r}\right).\end{aligned}$$

The interest rate on bonds and on capital needs to be the same. If this were not the case, households would only invest in bonds or capital, but not in both. In steady state, this interest rate is given by

$$r = \alpha A \bar{k}^{\alpha-1}.$$

This implies that the new steady state level of capital is given by the solution to

$$\beta \left[(1-\alpha) A \bar{k}^{\alpha} - n\bar{k} - \frac{b}{n} \right] = n\bar{k} + \frac{b}{\alpha A \bar{k}^{\alpha-1}}$$

which does not allow for a closed form solution.

To achieve the golden rule level of the capital stock, k_{GR}, note first that

$$r = \alpha A k_{GR}^{\alpha-1} = n.$$

Hence, the transfers to the young τ_1 are exactly zero. This implies that the level of b has to satisfy

$$\begin{aligned}
b &= n\left(\frac{\beta}{1+\beta}(1-\alpha)Ak_{GR}^{\alpha} - nk_{GR}\right) \\
&= nk_{GR}\left(\frac{\beta}{1+\beta}(1-\alpha)Ak_{GR}^{\alpha-1} - n\right) \\
&= nk_{GR}\left(\frac{\beta}{1+\beta}(1-\alpha)A\frac{n}{\alpha A} - n\right) \\
&= n^2 k_{GR}\left(\frac{\beta}{1+\beta}\frac{1-\alpha}{\alpha} - 1\right) \\
&= n^2 \left(\frac{\alpha A}{n}\right)^{\frac{1}{1+\alpha}} \left(\frac{\beta}{1+\beta}\frac{1-\alpha}{\alpha} - 1\right).
\end{aligned}$$

Suppose first, $b < 0$ or in other words the government <u>saves</u>. This is the case whenever there is underaccumulation. The household is indebted to the government and needs to cover this debt by additional savings in capital. To the contrary, suppose that $b > 0$ or in other words the government <u>is indebted</u>. This is the case whenever there is overaccumulation. The household has wealth in the form of holding government debt and reduces the investment into the capital stock.

Answer to Question 3:

1. Investors take the interest rate R as given. The supply of capital by investors to entrepreneurs is given by the solution to the problem

$$\max_{k^s \in [0,y]} Rk^s + \delta(y - k^s)$$

   Hence, for $R = \delta$, investors are indifferent between supplying any amount. For $R > \delta$ they supply all their funds y. Otherwise they do not supply anything.

2. Entrepreneurs take the interest rate R as given. The demand for capital by entrepreneurs solves the problem

$$\max_{k^d} 2\sqrt{k^d} - Rk^d$$

   so that the solution is given by

$$\frac{1}{\sqrt{k^d}} = R.$$

   Hence, the demand for capital is a decreasing function of the interest rate R.

3. Equating supply and demand for funds, we have that $k^s = k^d = y$ is the unique equilibrium. This is due to our assumption that for $k^d = y$, the interest rate R equals $\frac{1}{\sqrt{y}}$ and is larger than δ by assumption. For any interest rate lower than this amount, we would have that $k^d > y \geq k^s$. For any interest rate higher than this amount, we would have that $k^d < y = k^s$.

   Using $k^d = k^s = y$ and $R = 1/\sqrt{y}$ in the objective function for the investor and the entrepreneur, we find that consumption is given by $c_I = c_E = \sqrt{y}$.

4. The new problem is now given by

$$\max_{k^d} 2\sqrt{k^d} - Rk^d$$

   subject to

$$Rk^d \leq 2\rho\sqrt{k^d}$$

5. The first-order conditions for this problem are given by

$$\begin{aligned} \frac{1}{\sqrt{k^d}} - R + \lambda\left(\rho\frac{1}{\sqrt{k^d}} - R\right) &= 0 \\ \lambda\left(2\rho\sqrt{k^d} - Rk^d\right) &= 0 \end{aligned}$$

where λ is the Lagrange multiplier on the constraint and the second condition is the Kuhn-Tucker condition on the inequality constraint.

Suppose first that the constraint is not binding. Then, $2\rho\sqrt{k^d} > Rk^d$ and the Kuhn-Tucker condition implies then that $\lambda = 0$. Hence, we have that

$$R = \frac{1}{\sqrt{k^d}}.$$

Plugging this result in the constraint we find that the constraint does not bind if

$$\rho \geq 1/2.$$

Suppose next the constraint binds. Then $\lambda > 0$ and it must be the case from the Kuhn-Tucker condition that

$$R = \frac{2\rho}{\sqrt{k^d}}.$$

We know from the first first-order condition that whenever $\lambda > 0$, it must be the case that the MPK exceeds the interest rate or

$$f'(k^d) = \frac{1}{\sqrt{k^d}} > R = \frac{2\rho}{\sqrt{k^d}}.$$

This can only be the case when $\rho < 1/2$. Hence, the constraint is binding if and only if $\rho < 1/2$.

6. For $\rho = 1/2$, the constraint is not binding and we obtain that the equilibrium interest rate is given by $R = 1/\sqrt{y} = 2$.

For $\rho = 1/4$, we have that the constraint is binding. We check whether $y = k^s = k^d = 1/4$ – or full investment – is still an equilibrium. This is the case as long as there exists an interest rate $R \geq \delta$ with full investment and the constraint being binding. This is the case as long as

$$\frac{1}{4} = y = k^d = \left(\frac{2\rho}{R}\right)^2 \leq \left(\frac{2\rho}{\delta}\right)^2 = (2\rho)^2.$$

This shows that the parameter value $\rho = 1/4$ is precisely the critical value for which we have full investment of all funds with entrepreneurs, as $R = 1 = \delta$. Hence, for all $\rho \in [1/4, 1/2]$ there is full investment

For $\rho = 1/8$, we obtain that

$$y > \left(\frac{2\rho}{R}\right)^2$$

for any interest rate such that $R < \delta$. The market cannot clear at such interest rates, as the supply of capital k^s would be 0. Therefore, the interest rate satisfies $R = \delta = 1$

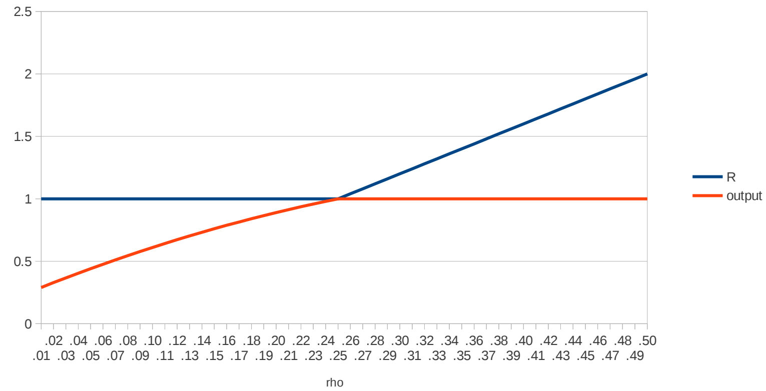
$$k^d = \left(\frac{2\rho}{\delta}\right)^2 = \frac{1}{16}.$$

This is an equilibrium, as investors are just indifferent between any amount to be supplied at δ. Note that the marginal product of capital exceeds the interest rate.

The first graph shows interest rates and output, while the second one compares consumption across investors and entrepreneurs. There are two regions: (i) $\rho \in [1/4, 1/2]$ and (ii) $\rho \in [0, 1/4]$. In the first region, the constraint is binding, but interest rates adjust so that all funds are still being lent to entrepreneurs. Output is constant at the optimal level. The variable ρ, however, influences interest rates and how output is split between investors and entrepreneurs. As ρ declines, so do interest rates and as a consequence, income is redistributed from investors to entrepreneurs. In the second region, output is not at the optimal level anymore. As ρ declines, less will be invested at the constant interest rate δ. This causes output to drop, with investors having a fixed level of consumption at $y = 1/4$, while entrepreneurs' consumption declines with ρ, eventually reaching 0. Finally, note that the drop in output (and consumption for entrepreneurs) is nonlinear as total output includes goods being stored so that it is given by $1/4 + 4\rho(1 - \rho)$.

7. Lowering the outside option δ will induce investors to put funds into entrepreneurs' projects at interest rates $\delta \leq R < 1$. To achieve that $k^s = y = 1/4$, interest rates have to fall sufficiently so that

$$k^d = y = \left(\frac{2\rho}{\delta}\right)^2$$



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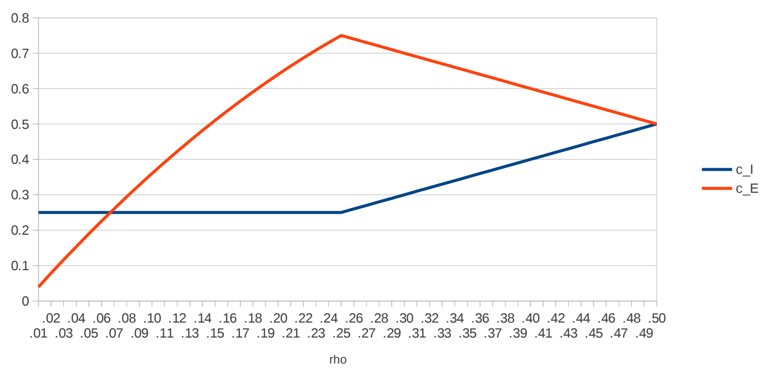
.01 .03 .05 .07 .09 .11 .13 .15 .17 .19 .21 .23 .25 .27 .29 .31 .33 .35 .37 .39 .41 .43 .45 .47 .49
rho
R
output



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.01 .03 .05 .07 .09 .11 .13 .15 .17 .19 .21 .23 .25 .27 .29 .31 .33 .35 .37 .39 .41 .43 .45 .47 .49
rho
c_I
c_E

or

$$\delta = \frac{2\rho}{\sqrt{y}} = \frac{1}{2}.$$

Note that total output is again at the first-best level of 1. But now, investors are worse off with consumption only at 1/8, while entrepreneur have even higher consumption than before.

One could (cautiously) interpret the change in δ as monetary policy lowering interest rates in response to a financial crisis where ρ falls sharply. The policy keeps output constant, but at the (private) "cost" of redistributing surplus from lenders to borrowers.

8. **Bonus:** One needs to show that total output is decreasing in δ for $\delta < 1$. Total output is given by

$$f(k^*) + \delta(y - k^*)$$

where

$$k^* = \left(\frac{2\rho}{\delta}\right)^{\frac{1}{1-\alpha}}$$

is the equilibrium amount of capital invested.

We need to show that

$$f'(k^*)\frac{\partial k^*}{\partial \delta} + (y - k^*) - \delta\frac{\partial k^*}{\partial \delta} < 0.$$

Taking derivatives and using the expression for k^*, this is the case if and only if

$$y\left(\frac{1-\alpha}{\alpha}\right) < k^*\left(\frac{1-\rho}{\rho}\right).$$

One can show that the right-hand side is a decreasing function of ρ whenever $\rho < \alpha$ (which is required for having a binding borrowing constraint). Finally, note that for $k^* = y$ and $\rho = \alpha$ the condition is fulfilled with equality.