**Economics 421** **Fall 2015**

**Topics in Macroeconomics** **Thorsten Koeppl**

## Answer Key for Assignment 2

### Answer to Question 1:

**Remark:** For the correct numerical answers, please set $\beta = 1$.

1. A stationary monetary equilibrium is determined by three equations: (i) the FONC that determines optimal decisions by the household; (ii) the feasibility constraint; and (iii) the stationarity condition imposed on market clearing for the money market that pins down the intertemporal ratio of prices $\frac{p_{t+1}}{p_t}$. These equations are

$$\begin{aligned} \frac{\frac{\partial u}{\partial c_2}}{\frac{\partial u}{\partial c_1}} &= \frac{p_{t+1}}{p_t} \\ c_1 + \frac{1}{n}c_2 &= y \\ \frac{M_t/p_t}{M_{t+1}/p_{t+1}} &= \frac{N_t(y-c_1)}{N_{t+1}(y-c_1)}. \end{aligned}$$

   The last equation establishes that $\frac{p_{t+1}}{p_t} = \frac{\mu}{n}$. Using the ratio of prices for a stationary equilibrium, the FONC then yields

$$\frac{\beta c_1}{c_2} = \frac{\mu}{n}$$

   and along with the feasibility constraint we find that the stationary equilibrium allocation is given by

$$\begin{aligned} c_1^* &= \left(\frac{\mu}{\mu+\beta}\right) y \\ c_2^* &= \left(\frac{n\beta}{\mu+\beta}\right) y \end{aligned}$$

   The initial price is determined by using the budget constraint of the initial old together with $m_0 = M_0/N_{-1}$

$$p_0 c_2 = m_0 = M_0$$

or

$$p_0 = \frac{M_0}{ny}\frac{\mu+\beta}{\beta}.$$

The price sequence given by

$$p_{t+1} = \left(\frac{\mu}{n}\right)^{t+1} p_0 = \left(\frac{\mu}{n}\right)^{t+1} \frac{M_0}{ny}\frac{(\mu+\beta)}{\beta},$$

since $p_{t+1} = \frac{\mu}{n} p_t$.

2. When the rate of increase in the money stock doubles the new rate of money growth is given by $2\mu$ instead of $\mu$. The stationary monetary equilibrium changes to

$$c_1^* = \left(\frac{2\mu}{2\mu+\beta}\right) y$$
$$c_2^* = \left(\frac{n\beta}{2\mu+\beta}\right) y$$
$$p_{t+1} = \left(\frac{2\mu}{n}\right)^{t+1} p_0$$

with $p_0 = \frac{M_0}{ny}\frac{2\mu+\beta}{\beta}$. Thus *money is not "superneutral"* as changing $\mu$ changes the equilibrium allocation. Note that the rate of inflation as defined by the ratio of prices also doubles or $\pi_t = \frac{p_{t+1}}{p_t} = \frac{2\mu}{n}$.

When the initial old have money equal to $2M_0$, but $\mu$ stays unchanged, the stationary allocation does not change either as the ratio of intertemporal prices does not change. However, the price level each period still changes as now $p_0 = \frac{2M_0}{ny}\frac{\mu+\beta}{\beta}$. Hence, *money is neutral* in the sense that a once-and-for-all change in money does not change the equilibrium allocation, but only the level of prices.

3. From the first part we know that to keep prices fixed, the central bank needs to choose $\mu = n$ such that $p_t = p_{t+1}$. The utility for each generation under such a monetary growth rule of $\mu = n$ is given by

$$u(c_t(t), c_t(t+1)) = \ln\left(\frac{ny}{n+\beta}\right) + \beta \ln\left(\frac{n\beta y}{n+\beta}\right)$$
$$u(c_{-1}(0)) = \ln\left(\frac{n\beta y}{n+\beta}\right).$$

An "optimal monetary growth rule" is a value for $\mu$ that is pareto-optimal for all generations but the initial old. This is the case for $\mu = 1$. The utility for each generation corresponding to such a rule is

$$u(c_t(t), c_t(t+1)) = \ln\left(\frac{y}{1+\beta}\right) + \beta \ln\left(\frac{n\beta y}{1+\beta}\right)$$
$$u(c_{-1}(0)) = \ln\left(\frac{n\beta y}{1+\beta}\right).$$

The initial old are better off with the optimal rule $\mu = 1$ whenever $n \geq 1$, but worse off whenever $n < 1$.

Comparing the utility for all other generations we get for the difference in utility

$$\begin{aligned}
\Delta(n,\beta) &= \ln\left(\frac{y}{1+\beta}\right) + \beta\ln\left(\frac{n\beta y}{1+\beta}\right) - \left(\ln\left(\frac{ny}{n+\beta}\right) + \beta\ln\left(\frac{n\beta y}{n+\beta}\right)\right) \\
&= \ln\left(\frac{y}{1+\beta}\frac{n+\beta}{ny}\right) + \beta\ln\left(\frac{n+\beta}{1+\beta}\right) \\
&= (1+\beta)\ln\left(\frac{n+\beta}{1+\beta}\right) - \ln n.
\end{aligned}$$

For $n = 1$, $\Delta = 0$. Taking a first-order derivative w.r.t. $n$, we obtain

$$\frac{\partial \Delta}{\partial n} = \frac{1+\beta}{n+\beta} - \frac{1}{n} = \frac{\beta(n-1)}{n(n+\beta)}.$$

Hence, the difference is strictly positive for all $n \neq 1$ independent of $\beta \in (0,1)$. Thus, we can conclude that $\mu = 1$ is preferred by all generations other than the initial old for all values of $n \in (0, \infty)$.

4. I will first solve the problem in full generality and then give a concrete example that facilitates the algebra. Note that we can normalize profits by the population size without influencing the results.

   The shaman solves

$$\max_{c_1,c_2} y - c_1 - c_2$$
$$\text{subject to}$$
$$\ln c_1 + \beta \ln c_2 \geq \bar{u}$$

where

$$\bar{u} = \ln\left(\frac{\mu}{\mu+\beta}y\right) + \beta\ln\left(\frac{\beta}{\mu+\beta}y\right)$$

is the utility a household gets in the perfect foresight equilibrium (PFE). The constraint must hold with equality. Hence, the Lagrangian gives

$$\frac{\beta c_1}{c_2} = 1.$$

This allows us to express total utility in terms of current consumption. Denote $\tilde{c}$ the consumption offered by the shaman. Then from the binding constraint using the FOC from the PFE we get

$$\ln\tilde{c}_1 + \beta\ln(\beta\tilde{c}_1) = \ln c_1 + \beta\ln\left(\frac{\beta}{2}c_1\right).$$

Rewriting this becomes

$$\begin{aligned}
\ln\left(\frac{\tilde{c}_1}{c}\right) &= \beta\ln\left(\frac{1}{2}\frac{c_1}{\tilde{c}_1}\right) \\
\frac{\tilde{c}_1}{c_1} &= \left(\frac{1}{2}\frac{c_1}{\tilde{c}_1}\right)^{\beta} \\
\tilde{c}_1 &= \left(\frac{1}{2}\right)^{\frac{\beta}{1+\beta}} c_1.
\end{aligned}$$

This describes the optimal transfer scheme offered by the shaman as

$$\tilde{c}_1 = \left(\frac{1}{2}\right)^{\frac{\beta}{1+\beta}} \left(\frac{\mu}{\mu+\beta}\right) y$$

$$\tilde{c}_2 = \beta\left(\frac{1}{2}\right)^{\frac{\beta}{1+\beta}} \left(\frac{\mu}{\mu+\beta}\right) y.$$

Let's do an example. Set $y = 1$ and $\beta = 1$. Then, the utility from the PFE is given by

$$\ln 2/3 + \ln 1/3 = \ln 2/9.$$

We know from the FOC from the shaman's problem that

$$\tilde{c}_1 = \tilde{c}_2.$$

Hence, from the binding participation constraint we have

$$2\ln\tilde{c}_1 = \ln 2/9$$

or

$$\tilde{c}_1 = \tilde{c}_2 = \frac{\sqrt{2}}{3}.$$

The old will prefer the shaman's proposal as their consumption increases. This can be seen by

$$\tilde{c}_2 = \beta \tilde{c}_1 = \beta \left( \frac{\beta}{2} \right)^{\frac{\beta}{1+\beta}} c_1 > \beta \frac{1}{2} c_1 = c_2.$$

This is clear in the example since $1/3 < \sqrt{2}/3$.

The intuition for this result is that the shaman can offer less distortions to the young generation by smoothing consumption better across time through the transfer scheme. This involves that $c_1$ decreases and $c_2$ increases. This allows the shaman to extract part of the endowment, as he reduces the distortion introduced by the inefficient monetary policy. Still, all generations are at least as well off with the scheme, but the initial old are strictly better off as they only care about $c_2$ which has increased. Hence, a shaman could offer a better system of transfers than monetary policy at $\mu = 2$.

**Answer to Question 2:**

1. The stationary PFE is given by a solution to the following four equations

$$\begin{aligned} \sqrt{\frac{c_1}{c_2}} &= \frac{p_{t+1}}{p_t} \\ \frac{p_{t+1}}{p_t} &= \mu \\ c_1 + c_2 &= 1 - g \\ g &= \left(1 - \frac{1}{\mu}\right)(1 - c_1), \end{aligned}$$

   where all variables have been expressed in per capital terms.

   Hence, we have that

$$c_2 = \frac{1}{\mu^2} c_1$$

   and using this result and the expression for $g$ in the market clearing condition we obtain

$$c_1 = \frac{\mu}{\mu + 1}.$$

   It follows immediately that

$$\begin{aligned} c_2 &= \frac{1}{\mu(\mu+1)} \\ g &= \frac{\mu - 1}{\mu(\mu+1)}. \end{aligned}$$

   Finally, prices are given by

$$p_t = \mu^t p_0,$$

   with $p_0 = \frac{M_0}{N_{-1}} \mu(\mu+1) = \frac{M_0}{2} \mu(\mu+1)$.

2. The graph is shown and explained in the answer to part (e).

3. To maximize government seignorage, we solve

$$\max_{\mu} \left(1 - \frac{1}{\mu}\right)(1 - c_1(\mu))$$

   subject to

$$c_1(\mu) = \frac{\mu}{\mu + 1}.$$

The constraint expresses the fact that households increase their consumption in period 1 in response to higher money growth in a PFE. Plugging the constraint into the objective function and taking a derivative with respect to $\mu$, we obtain

$$\frac{\partial g}{\partial \mu} = \frac{1}{(\mu(\mu+1))^2}\left(-\mu^2 + 2\mu + 1\right).$$

Setting this expression to 0 to find an extremum, we find that the relevant root for this quadratic expression is given by

$$\mu = 1 + \sqrt{2}.$$

4. For a temporary equilibrium in period $t$ we have that the state variables are given by the money holdings of the old generation, $m_{t-1}$, and the expected prices $p_{t+1}^e$. Note that

$$m_{t-1} = \frac{1}{\mu}\frac{M_t}{N_t} = \frac{1}{\mu}m_t.$$

From the household's optimization problem we obtain the intertemporal first-order condition

$$\sqrt{\frac{c_t(t)}{c_t(t+1)}} = \frac{p_{t+1}^e}{p_t} = 1,$$

since households *always* expect prices to stay constant. Hence, we have $c_t(t) = c_t(t+1)$ and from the intertemporal budget constraint we find that

$$\begin{aligned} c_t(t) + \frac{p_{t+1}^e}{p_t}c_t(t+1) &= 1 \\ c_t(t) + c_t(t) &= 1 \\ c_t(t) &= 1/2. \end{aligned}$$

Government's seignorage as a function of prices in period $t$ is given by

$$g_t = \frac{\Delta m_t}{p_t} = \left(\frac{\mu-1}{\mu}\right)\frac{m_t}{p_t}.$$

Generation $t-1$'s consumption in period $t$ is given by

$$c_{t-1}(t) = \frac{m_{t-1}}{p_t} = \frac{1}{\mu}\frac{m_t}{p_t}.$$

From the market clearing condition we can calculate the temporary equilibrium price as follows

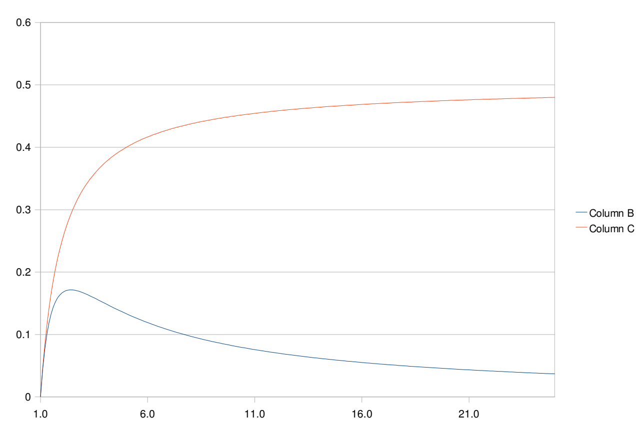
$$\begin{aligned} c_t(t) + c_{t-1}(t) + g_t &= 1 \\ 1/2 + \frac{1}{\mu}\frac{m_t}{p_t} + \left(\frac{\mu-1}{\mu}\right)\frac{m_t}{p_t} &= 1 \\ \frac{m_t}{p_t} = 1/2. \end{aligned}$$

Hence, the equilibrium allocation is given by

$$\begin{aligned} c_t(t) &= 1/2 \\ c_{t-1}(t) &= \frac{1}{2\mu} \\ g_t &= 1/2\left(\frac{\mu-1}{\mu}\right). \end{aligned}$$

Prices are given by $p_t = \mu^t p_0$, with $p_0 = \frac{M_0}{N_{-1}} 2\mu = M_0\mu$.

5. The graph below shows seignorage as a function of money growth.



In the PFE a Laffer-curve arises and seignorage converges to 0 when $\mu \to \infty$. Why? Household's will shift consumption more and more to the first period, i.e. $c_1 \to 1$ and $c_2 \to 0$ as $\mu \to \infty$.

In the Temporary Equilibrium we have considered, households mistakenly do not take into account that prices will rise between periods. Indeed, when they expect prices to stay constant, they do not change their consumption in the first period as the rate of money growth $\mu$ changes, i.e. $c_t(t) = 1/2$ independent of $\mu$. As a consequence, seignorage is strictly increasing in the rate of money growth, or $g \to 1/2$ as $\mu \to \infty$.

**Answer to Question 3:**

1. With debt financing only, the balanced budget condition in period $t$ is given by

$$\begin{aligned} N(t)\frac{b(t)}{1+r(t)} &= N(t-1)b(t-1) \\ \frac{b(t)}{1+r(t)} &= \frac{b(t-1)}{n}. \end{aligned}$$

2. The household's problem is given by

$$\begin{aligned} &\max_{c_t(t),c_t(t+1),b(t)} \ln c_t(t) + \ln c_t(t+1) \\ &\text{subject to} \\ &c_t(t) + \frac{b(t)}{1+r(t)} = y_1 \\ &c_t(t+1) = y_2 + b(t) \end{aligned}$$

where the household takes the interest rate $r(t)$ as given.

The FOC is given by

$$\frac{c_t(t+1)}{c_t(t)} = 1 + r(t).$$

3. The intertemporal budget constraint is given by

$$c_1 + \frac{1}{1+r(t)}c_2 = y_1 + \frac{1}{1+r(t)}y_2.$$

Market clearing yields

$$c_1 + \frac{1}{n}c_2 = y_1 + \frac{1}{n}y_2$$

so that interest rate have to be constant in a stationary perfect foresight equilibrium. Hence, $1 + r(t) = n = 2$ for all $t$.

The equilibrium consumption allocation is thus given by

$$\begin{aligned} c_1 &= \frac{2n+1}{2n} = \frac{5}{4} \\ c_2 &= \frac{2n+1}{2} = \frac{5}{2}. \end{aligned}$$

The initial debt level consistent with this consumption allocation is

$$b^{SS}(-1) = c_{-1}(0) - y_2 = \frac{2n-1}{2} = \frac{3}{2}.$$

From the government budget constraint, it follows that the sequence of per capita debt levels is thus

$$b^{SS}(t) = b^{SS}(-1) = \frac{3}{2}.$$

The total amount of debt $N(t)b^{SS}(t)$ in the economy, however, increases at rate $n$.

An important remark. Debt works just like an initial amount of money here – which is really non-interest bearing "debt". One could transfer this economy into a monetary one with a fixed supply of money that's initially given to the old generation. All what is necessary is a vehicle for savings here – whether it is money in constant supply or total debt increasing at the rate of population growth. Note that intertemporal prices (or interest rates) are fixed at $n$ in both cases. This also clarifies that $p_{t+1}/p_t$ is simply a (nominal) interest rate.

4. To find the lump-sum scheme we just have to look at the budget constraints of the household,

$$\begin{aligned} \tau_1(t) &= y_1 - c_1 = \frac{2n-1}{2n} = \frac{3}{4} \\ \tau_2(t) &= y_2 - c_2 = \frac{1-2n}{2} = -\frac{3}{2}. \end{aligned}$$

It is straightforward to check that these lump-sum transfers satisfy the government budget constraint

$$N(t)\tau_1(t) + N(t-1)\tau_2(t) = 0.$$

5 & 6. I will first describe the algorithm.

We have $b(-1) = 1.01 b^{SS}(-1) = 1.515$. The budget constraint of the initial old yields

$$c_{-1}(0) = y_2 + b(-1).$$

For the first iteration, we have $c_{-1}(0) = 2.515$ as the initial condition.

Step 1: Market clearing gives

$$c_t(t) = y_1 + \frac{1}{n}\left(y_2 - c_{t-1}(t)\right).$$

Note that we know $c_{t-1}(t)$. For the first iteration this gives us $c_0(0) = 1.2425$.

Step 2: Solve the household problem given by

$$\begin{aligned} \frac{c_t(t+1)}{c_t(t)} &= 1 + r(t) \\ c_t(t+1) &= y_2 + b(t) = y_2 + b(t-1)\frac{1+r(t)}{n}. \end{aligned}$$

Since we know $c_t(t)$ from Step 1 and $b(t-1)$ from the previous iteration, this can be solved for $c_t(t+1)$ and $1+r(t)$. For the first iteration, this yields

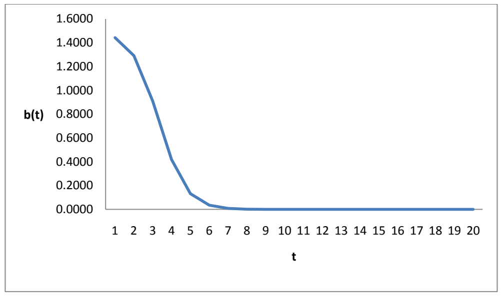
$$\begin{aligned} c_0(1) &= 2.5619 \\ 1 + r(0) &= 2.0619. \end{aligned}$$

Step 3: Calculate $b(t) = b(t-1)\frac{1+r(t)}{n}$.

Step 4: Go back to Step 1 with $c_t(t+1)$ and repeat the steps for period $t+1$. In the first iteration, we obtain $b(0) = 1.5619$.

The first set of graphs shows the level of debt level and interest rates over time. The economy converges towards autarky, with ever decreasing debt levels and interest rates. In autarky, the debt is zero, but (gross) interest rates are at 0.5. This implies that the policy reduces welfare for all generations. In fact, (net) interest rates need to become negative to keep people from saving (-50%). The last two exercises imply that there is the notion of an optimal debt level for this economy given by $b^{SS}$.

The second set of graphs shows the evolution of debt and interest rates. Debt grows exponentially here, as we start off with a debt level above $b^{SS}$. In this example, it turns out that after period 2, the debt level is so high that the numbers do not make sense anymore. Debt above the steady state level is not feasible. Note that the allocation gets more and more distorted to the future. Hence, an equilibrium does not exist in any economy with $b(-1) > b^{SS}$.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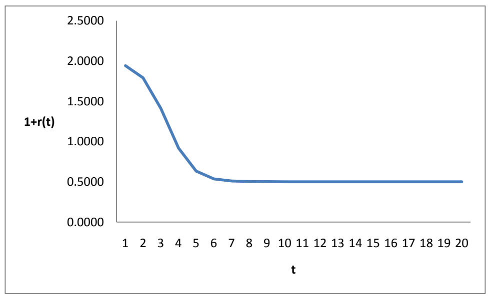
t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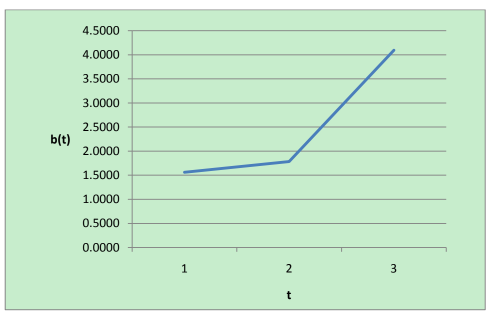
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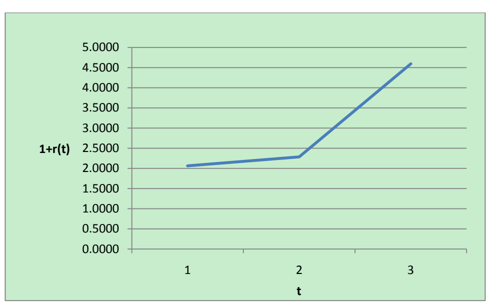
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