Economics 421 **Winter 2015**

Topics in Macroeconomics **Thorsten Koeppl**

Answer Key for Assignment 1

Answer to Question 1:

(a) We do not assume stationary allocations here and derive the general case. Feasibility in period t requires

$$N_t c_t(t) + N_{t-1} c_{t-1}(t) \leq N_t y.$$

Using the fact that $N_t = n N_{t-1}$ we have that

$$c_t(t) + \frac{1}{n} c_{t-1}(t) \leq y.$$

The diagram is the same as shown in the lecture notes.

(b) Consider the stationary allocation $(c_1, c_2) = (\frac{3}{4}y, \frac{1}{2}y)$. The allocation is feasible for $n = 2$, since

$$\frac{3}{4}y + \frac{1}{2}\frac{1}{2}y = y.$$

However, the allocation $(\tilde{c}_1, \tilde{c}_2) = (\frac{1}{4}y, \frac{3}{2}y)$ is also feasible ($\frac{1}{4}y + \frac{3}{2}\frac{1}{2}y = y$) and delivers strictly more utility to *all* generations (and not only the intial old) than the allocation (c_1, c_2). Hence, the allocation (c_1, c_2) is not Pareto-optimal.

With a stationary allocation, this can also be seen by comparing the intertemporal marginal rate of substitution with the (negative) slope of the feasibility equation, where the later has to be smaller. Here, we have $-\sqrt{\frac{2}{3}} > -2$, which shows that (c_1, c_2) is not Pareto-optimal.

(c) We first find the Pareto-optimal allocation that is most preferred by all generations (save the initial old). It solves the problem

$$\max_{c_1, c_2} \sqrt{c_1} + \sqrt{c_2}$$

subject to

$$c_1 + \frac{c_2}{n} = y$$

The FOC for this problem is given by

$$\sqrt{\frac{c_1}{c_2}} = n^2.$$

Using the feasibility condition with $n = 2$, the solution is thus given by

$$\begin{aligned} c_1 &= \frac{1}{3}y \\ c_2 &= \frac{4}{3}y. \end{aligned}$$

This is just the point "A" in our diagram in the lecture. Hence, every other point on the boundary of the feasible set with $c_1 < \frac{1}{3}y$ is also Pareto-optimal. More formally the set of all Pareto-optimal allocations is given by

$$\mathcal{PO} = \left\{ (c_1, c_2) | 0 \leq c_1 < \frac{1}{3}y \wedge c_1 + \frac{c_2}{n} = y \right\}.$$

(d) Since there cannot be any trade, the young and the old simply eat whatever resources they have or

$$\begin{aligned} c_t(t) &\leq y + \tau_t(t) \\ c_{t-1}(t) &\leq \tau_{t-1}(t). \end{aligned}$$

Hence, to achieve the Pareto efficient allocation that we have found in part (d), transfers for the young and the old need to be

$$\begin{aligned} \tau_1 &= c_1 - y = -\frac{2}{3}y \\ \tau_2 &= c_2 = \frac{4}{3}y. \end{aligned}$$

Note that this transfer scheme is feasible, since

$$N_t\tau_1 + N_{t-1}\tau_2 = N_{t-1}y\left(-n\frac{2}{3} + \frac{4}{3}y\right) = 0.$$

In general, given individual endowments, there exists a feasible transfer scheme such that we can achieve any Pareto efficient allocation.

(e) Each generation's maximization problem is now given by

$$\max_{c_t(t),c_{t-1}(t),s_t} \sqrt{c_t(t)} + \sqrt{c_t(t+1)}$$

subject to

$$c_t(t) + s_t \leq y$$

$$c_{t-1}(t) \leq r s_t.$$

Here r is the gross return, so that for $r = 1$, one would just get the initial investment back and for $r > 1$ ($r < 1$) one would make a positive (negative) return.

The solution for this problem is given by the FONC and the life-time budget constraint

$$\begin{aligned} \frac{c_t(t+1)}{c_t(t)} &= r^2 \\ c_t(t) + \frac{1}{r} c_t(t+1) &= y \end{aligned}$$

which yields the stationary allocation $(c_1, c_2) = (1/(1+r)y, r^2/(1+r)y)$.

Comparing the utilities for each generation $t \geq 0$, storage dominates the transfer scheme if and only if

$$\begin{aligned} \sqrt{\frac{1}{1+r}y} + \sqrt{\frac{r^2}{1+r}y} &\geq \sqrt{\frac{1}{1+n}y} + \sqrt{\frac{n^2}{1+n}y} \\ \sqrt{1+r} &\geq \sqrt{1+n} \\ r &\geq n. \end{aligned}$$

(f) If the end of the transfer scheme is publicly announced, it would unravel backwards from period T. The young of generation t have a cost in terms of the transfer $\tau_1 < 0$, but no benefit. Hence, they would prefer putting all their resources into storage. This implies, however, that the young of generation $T - 1$ cannot be promised a transfer $\tau_2 > 0$ anymore when they are old. This logic continues until generation 0.

Suppose now that the end of the transfer scheme comes at a complete surprise at T. If the end of the scheme is announced before consumption takes place in period T, the young simply save and we get the solution in part (e). The old of generation $T-1$ would end up with zero consumption in period T. If the end is announced after consumption takes place, the young of generation T bear all the costs.

Remark: As an intermediate case, one could consider that the end is announced before consumption takes place, but after transfers have been made. The young of generation T would then choose their investment into storage optimally, given that their endowment would now be only $y - \tau_1 = 1/3y$. After period T, we would be back to part (e) for the optimal investment behavior.

Answer to Question 2:

(a) A *stationary (monetary) equilibrium with perfect foresight* with an amount M of money is given by a sequence of prices $\{p(t)\}_{t=0}^{\infty}$, expected prices $\{p^e(t+1)\}_{t=0}^{\infty}$, money demands $\{m_t\}_{t=0}^{\infty}$ and an allocation $\{c_t(t), c_t(t+1)\}_{t=0}^{\infty}$ and $c_{-1}(0)$ such that

(i) the allocation maximizes utility for each generation taking prices and expected prices as given

(ii) the allocation is feasible (or, equivalently, markets clear), i.e,

$$c_t(t) + c_{t-1}(t) = y$$

(iii) the allocation is stationary, i.e.,

$$c_t(t) = c_1 \quad \text{for all } t \geq 0$$
$$c_t(t+1) = c_2 \quad \text{for all } t \geq -1$$

(iv) household perfectly forecast future prices, i.e. $p^e(t+1) = p(t+1)$ for all t.

(b) With perfect foresight, we can write the budget constraints for generation $t \geq 0$ as

$$p(t)c_t(t) + m_t \leq p(t)y$$
$$p(t+1)c_t(t+1) \leq m_t.$$

Since the utility function is strictly increasing in both arguments, we must have that both equations hold with equality. Rearranging the equations and substituting for $c_t(t)$ and $c_t(t+1)$ in the utility function, we obtain

$$\ln\left(y - \frac{m_t}{p(t)}\right) + \beta \ln\left(\frac{m_t}{p(t+1)}\right)$$

for the objective function. The FONC are then given by

$$\frac{p(t+1)}{p(t)} = \beta \frac{y - \frac{m_t}{p(t)}}{\frac{m_t}{p(t+1)}}.$$

Rearranging, we obtain for the money demand equation

$$\frac{m_t}{p(t)} = y\frac{\beta}{1+\beta}.$$

(c) For the stationary equilibrium

$$\beta \frac{c_1}{c_2} = \frac{p(t+1)}{p(t)} = 1$$

This implies $\beta c_1 = c_2$. Combining with the market clearing condition, $c_1 + c_2 = y$, we have

$$c_1 = \frac{1}{1+\beta} y$$
$$c_2 = \frac{\beta}{1+\beta} y$$

From the budget constraint of the initial old generation, $p(0)c_2 = M$, we have

$$p(0) = \frac{M}{y} \frac{1+\beta}{\beta}.$$

(d) It is straightforward to check that the real allocation (c_1, c_2) – and, hence, savings – is not affected by the change in the initial stock of money. Money is neutral, so that only the price level changes 1-1 with the change in the money stock M.

Answer to Question 3:

(a) The per capita endowments are given by

$$\begin{aligned} m_{-1} &= \frac{M}{N_{-1}} = \frac{M}{2} \\ y &= \frac{Y}{N_{-1}} = \frac{Y}{2}. \end{aligned}$$

(b) In a stationary monetary equilibrium with perfect foresight, prices have to be constant, since

$$\frac{p(t+1)}{p(t)} = \frac{1}{n} = 1.$$

Given prices, households chose their consumption according to the FONC

$$\frac{p(t+1)}{p(t)} = \beta\sqrt{\frac{c_t(t)}{c_t(t+1)}}.$$

With stationarity, this implies that

$$c_2 = \beta^2 c_1.$$

Using market clearing, $c_1 + c_2 = Y/2$, we find that the stationary equilibrium allocation is given by

$$c_1 = \frac{Y}{2}\frac{1}{1+\beta^2} \tag{0.1}$$

$$c_2 = \frac{Y}{2}\frac{\beta^2}{1+\beta^2}. \tag{0.2}$$

For a stationary allocation, we also have to ensure that $c_{-1}(0) = c_2$. Using the budget constraint for the initial old generation we get

$$p(0)c_2 = \frac{M}{2}$$

or

$$p(0) = \frac{M}{2}\frac{2}{Y}\frac{1+\beta^2}{\beta^2} = \frac{M}{Y}\left(1 + \frac{1}{\beta^2}\right).$$

Hence (c_1, c_2), $m_t = M/2$ and $p(t) = p(0)$ for all t describe a stationary monetary equilibrium.

(c) From the household's optimization problem we have that

$$\begin{aligned} \frac{p^e(t+1)}{p(t)} &= \beta\sqrt{\frac{c_t(t)}{c_t^e(t+1)}} \\ p(t)c_t(t) + p^e(t+1)c_t^e(t+1) &= p(t)\frac{Y}{2}. \end{aligned}$$

The first equation is the FONC and the second one the intertemporal budget constraint. Prices are now given by

$$\frac{p^e(t+1)}{p(t)} = a.$$

This implies that we have

$$\begin{aligned} c_t^e(t+1) &= \left(\frac{\beta}{a}\right)^2 c_t(t) \\ c_t(t) + ac_t^e(t+1) &= \frac{Y}{2}. \end{aligned}$$

Hence, we have $c_t(t) = \frac{Y}{2}\frac{a}{a+\beta^2}$ and $c_t^e(t+1) = \frac{Y}{2}\frac{\beta^2}{a^2+a\beta^2}$.

(d) The actual consumption in period $t+1$ for generation t is given by

$$c_t(t+1) = \frac{M}{2p(t+1)},$$

where the money demand follows directly from money market clearing.

(e) We only are left to find prices. To do so we use the market clearing condition in period t which is given by

$$c_t(t) + c_{t-1}(t) = \frac{Y}{2}.$$

We can use the actual consumption of generation $t-1$ in t to obtain

$$\begin{aligned} \frac{Y}{2}\frac{a}{a+\beta^2} + \frac{M}{2p(t)} &= \frac{Y}{2} \\ \frac{M}{2p(t)} &= \frac{Y}{2}\left(1 - \frac{a}{a+\beta^2}\right) \\ p(t) &= \frac{M}{Y}\frac{a+\beta^2}{\beta^2}. \end{aligned}$$

Hence, prices are constant, but potentially at a different level than in the perfect foresight equilibrium. The temporary equilibrium is thus given by such prices and the allocation

$$\begin{aligned} c_t(t) &= \frac{Y}{2}\frac{a}{a+\beta^2} \\ c_t(t+1) &= \frac{Y}{2}\frac{\beta^2}{a+\beta^2}. \end{aligned}$$

(f) If $a = 1$, we have a perfect foresight equilibrium where households forecast future prices correctly. If $a < 1$ ($a > 1$), households expect deflation (inflation), i.e. decreasing (increasing) prices over time, $p^e(t+1) < p(t)$ ($p^e(t+1) > p(t)$). Consumption when young is increasing in a, since

$$\frac{\partial c_t(t)}{\partial a} = \frac{Y}{2}\left(\frac{\beta}{a+\beta^2}\right)^2 > 0.$$

Hence, consumption when young falls (rises) with deflation (inflation) relative to the perfect foresight equilibrium. Consumption when old reacts in the opposite way. The intuition is simple. Deflation ($a < 1$) implies a return bigger than 1 on money inducing households to save more. Inflation ($a > 1$) implies a return less than 1 on money with households saving less.

Note that the perfect foresight allocation is pareto-optimal. Also, all temporary allocations fall on the boundary of the feasible set. When $a > 1$, welfare for all generations is lower as second period consumption falls relative to the perfect foresight equilibrium. When $a < 1$, we still have a pareto-optimal allocation. However, the initial old generation gains from deflationary expectations, while all other generations are worse off.