ECON 421

Steady State and Dynamic Inefficiency

Fall 2015

Steady State

From the savings function, we have that

$$k(t+1) = \left(\frac{1}{n}\frac{\beta}{1+\beta}(1-\alpha)A(0)\right)(1+\gamma)^t\, k(t)^\alpha$$

Assume $A(t) = A$ and log utility.

In per capita terms, the steady state level of capital is thus given by

$$\bar{k} = \left(\frac{1}{n}\frac{\beta}{1+\beta}(1-\alpha)A\right)^{\frac{1}{1-\alpha}}$$

The overall capital stock is growing in steady state at the fixed growth rate

$$\frac{K(t+1)}{K(t)} = \frac{\bar{k}N(t+1)}{\bar{k}N(t)} = n$$

Balanced Growth Path

Assume $A(t) = (1+\gamma)A(t-1)$.

With log utility, we obtain the first-order difference equation

$$\begin{aligned}
k(1) &= \frac{1}{n}\frac{\beta}{1+\beta}w(t) \\
k(1) &= \left(\frac{1}{n}\frac{\beta}{1+\beta}(1-\alpha)A(0)\right)k(0)^{\alpha} \\
k(2) &= \left(\frac{1}{n}\frac{\beta}{1+\beta}(1-\alpha)A(0)\right)(1+\gamma)\,k(1)^{\alpha} \\
k(t+1) &= \left(\frac{1}{n}\frac{\beta}{1+\beta}(1-\alpha)A(0)\right)(1+\gamma)^{t}\,k(t)^{\alpha}
\end{aligned}$$

Hence:

$$k(t+1) = \kappa(1+\gamma)^{t}k(t)^{\alpha}$$

Thus we obtain for the growth rate of capital

$$\frac{k(t+1)}{k(t)} = \frac{(1+\gamma)^t k(t)^\alpha}{(1+\gamma)^{t-1} k(t-1)^\alpha}$$

or in the long-run

$$\lim_{t\to\infty} \frac{k(t+1)}{k(t)} = (1+\gamma) \lim_{t\to\infty} \left(\frac{k(t)}{k(t-1)} \right)^\alpha$$

This implies that the **growth rate in the economy** $1+\tilde{\gamma}$ is given by

$$1+\tilde{\gamma} = 1+\gamma_k = (1+\gamma)^{\frac{1}{1-\alpha}} .$$

At a balanced growth path, we require that per capita capital, output and consumption grow at the same rate.

For output, we have

$$\begin{aligned}
\frac{y(t+1)}{y(t)} &= (1+\gamma)\left(\frac{k(t+1)}{k(t)}\right)^{\alpha} \\
\lim_{t\to\infty}\frac{y(t+1)}{y(t)} &= (1+\gamma)\lim_{t\to\infty}\left(\frac{k(t+1)}{k(t)}\right)^{\alpha} \\
1+\gamma_y &= (1+\gamma)^{\frac{1}{1-\alpha}}
\end{aligned}$$

For consumption, we have

$$\begin{aligned}
\frac{c_{t+1}(t+1)}{c_t(t)} &= \frac{w(t+1)}{w(t)} \\
\lim_{t\to\infty}\frac{c_t(t+1)}{c_t(t)} &= \lim_{t\to\infty}\left(\frac{k(t+1)}{k(t)}\right)^{\alpha} \\
1+\gamma_c &= (1+\gamma)^{\frac{1}{1-\alpha}}
\end{aligned}$$

Hence:

$$1+\gamma_k = 1+\gamma_c = 1+\gamma_y = (1+\gamma)^{\frac{1}{1-\alpha}} = 1+\tilde{\gamma}$$

Long-run Real Interest Rates

We detrend variables by $(1+\tilde{\gamma})$. This yields for the law of motion of capital

$$\begin{aligned}
(1+\tilde{\gamma})\tilde{k}(t+1) &= \frac{(1+\tilde{\gamma})^{t+1}}{(1+\tilde{\gamma})^t}\frac{k(t+1)}{(1+\tilde{\gamma})^{t+1}} = \frac{1}{n}\frac{s(t+1)}{(1+\tilde{\gamma})^t} \\
&= \left(\frac{1}{n}\frac{\beta}{1+\beta}(1-\alpha)A(0)(1+\gamma)^t\right)\frac{k(t)^\alpha}{(1+\tilde{\gamma})^t} \\
&= \left(\frac{1}{n}\frac{\beta}{1+\beta}(1-\alpha)A(0)\right)\tilde{k}(t)^\alpha
\end{aligned}$$

Interest rates in steady state are given by

$$\bar{r} = \alpha A(0)\bar{\tilde{k}}^{\alpha-1} = n(1+\tilde{\gamma})\frac{\alpha}{1-\alpha}\frac{1+\beta}{\beta}$$

Conclusion: Interest rates depend positively on population and productivity growth, but negatively on the discount factor.

Optimality of the Steady State

The steady state satisfies three equations:

$$\begin{aligned} \frac{\beta u'(c_2)}{u'(c_1)} &= \frac{1}{r(\bar{k})} \\ \bar{k} &= \frac{1}{n} s\left(w(\bar{k}), r(\bar{k})\right) \\ A\bar{k}^{\alpha} &= s\left(w(\bar{k}), r(\bar{k})\right) + c_1 + \frac{1}{n} c_2 \end{aligned}$$

Is the SS eql. optimal?

We consider the “Golden rule” allocation.

$$\max_{c_1, c_2, \bar{k}} u(c_1, c_2)$$

subject to

$$A\bar{k}^{\alpha} = n\bar{k} + c_1 + \frac{1}{n} c_2$$

Consider first **productive efficiency**.

$$\max_k \phi(k) = \max_k Ak^\alpha - nk$$

Solution – Golden Rule capital stock

$$\alpha A k_{GR}^{\alpha-1} = n$$

Given k_{GR}, **allocative efficiency** requires

$$\begin{aligned} \frac{\beta u'(c_2)}{u'(c_1)} &= \frac{1}{n} \\ c_1 + \frac{1}{n}c_2 &= \phi(k_{GR}) \end{aligned}$$

ADD GRAPH

Externality on Savings Decisions

In general, we have that

$$s\left(w(k_{GR}), n\right) \neq nk_{GR}$$

Individual savings decisions will not lead to socially optimal capital accumulation.

Why?

Generation t does not take into account the impact of its savings decision on future generations.

This is once again related to the problem where we compare the MRT r associated with a storage technology and the MRT of transfers which is linked to the population growth rate n.

Two possibilities.

1) Overaccumulation: $\phi'(\bar{k}) < 0$

- At some stage, $k_t > k_{GR}$, as $k_t \to \bar{k}$.
- One could increase consumption for all generations after t by reducing capital accumulation.
- How? Levy a lump-sum taxes on the returns of savings.
- This is related to PAYG pension systems.

2) Underaccumulation: $\phi'(\bar{k}) > 0$

- One needs to increase capital accumulation.
- This implies a decrease in consumption.
- Hence, there is a conflict of interest.