The Impact of Non-Practising Entities on Investment in R&D

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Abstract

In studying the impact of non-practising-entities on investment and innovation much of the recent focus has been on the role of rent-seeking behavior through the court system — as such entities seek to establish and enforce ownership rights on intellectual property. As a result, issues relating to behavior within the context of the legal framework have received significant attention. This paper considers the impact of NPE's from a different perspective where the validity and assignment of intellectual property rights is unequivocal, and in this context examines the economic contribution of an NPE to investment in R&D.

1 Introduction

The term 'non-practising-entity' (NPE) is used to denote an intellectual property holder that obtains revenue from the licensing of its intellectual property, and not directly from its use. This term covers a heterogeneous group of patent holders including universities, individual inventors, patent holding companies and so on — in principle, any entity holding a patent that is not practising or producing on the basis of the patent. Because an NPE does not produce, revenue must be obtained from license fees. This in turn often requires that the NPE assert patent infringement against an operating firm as part of the process of achieving a licensing agreement. As a result, the practical and strategic issues involved tend to be examined from a legal perspective (see, for example [3, 4, 5, 10, 11, 13]). Since litigation is expensive and the outcome unpredictable, the risks surrounding litigation are large and have significant influence on the behavior of all parties — in terms of incentives to initiate a lawsuit, to defend, to settle, to appeal, and so on. These are major institutional considerations in assessing the role of the NPE's in promoting innovation since they raise issues regarding rent-seeking, innovation holdup and so on.

Such issues arise because the exact status of an infringement claim cannot be immediately verified. The actual research contributions of an NPE are difficult to assess and the rewards obtained by the NPE reflect not only the research contribution but also the vagaries of the litigation process. Thus, the merits or otherwise of the NPE are conflated with other issues. This paper puts these matters aside, considering an environment where rights issues are fully resolved, and examines the advantages or disadvantages generated by NPE's in terms of innovation when the innovation contribution can be correctly measured.

The paper focuses on two cases in detail, where an NPE operates with either one or two firms (with most emphasis on the two firm case). Initially, the case of monopoly is considered: this motivates the general discussion and provides some of the notation for the two-firm case. In each case, the paper examines the impact of an NPE on the volume of investment and the quality of innovation. Even in the simplest circumstances, it turns out to be surprisingly difficult to give unambiguous answers regarding the merits of NPE's.

An outline of the paper is as follows. Section (2) introduces the model, then sections (3) and (4) considers the single and two firm cases respectively. (Licensing plays a central role and is discussed in detail in section (4.1).) In both cases, equilibrium in a benchmark model without an NPE is compared with equilibrium in the presence of an NPE. This is considered in section (4.4). Section (5) considers the linear case. A full characterization of equilibrium behavior is given. While a prevalent view of the NPE is that of a rent-seeker, the results in this paper suggest a far more ambiguous assessment. Specifically, if the NPE is a good-faith investor (investing in exactly the same manner as operating firms), then for some regions of cost the presence of an NPE may raise overall investment and potentially the overall level of innovation. The exact circumstances under which this may be true depend on a range of (endogenously determined) threshold parameters in the model.

2 The Model.

There are n firms ($n \in \{1, 2\}$) and a non-practising entity (NPE). Thus, the NPE operates alongside a monopoly or the NPE operates alongside two firms operating in differentiated markets. Innovation centers on cost reduction in production: a firm or NPE investing in innovation develops a cost reducing innovation. Initial cost is $c = 1$, and with investment of ρ in R&D, a firm draws a new cost technology x from $[0, 1]$

according a cumulative distribution F while the NPE draws the innovation according to the distribution G. The patent winner is the party that achieves the best innovation (the smallest value of x), and the winner is assigned a patent such that without ownership or licensing of the patent, a firm would remain with the cost structure $c = 1$. Assuming the distributions F and G are continuous, there is 0 probability of a tie for lowest value. For the case of monopoly, the firm's profit is assumed to be a function of the innovation quality, $\pi(x)$, in the two firm case, profit for each firm is a function of both firms cost parameter. In both cases, profit is derived from underlying demand so that royalty fees on quantity may be defined.

3 Monopoly

Let market demand be $p(q)$ with corresponding profit $\pi(x) = \max\{p(q)q - xq\}$, where x is constant marginal cost. With no NPE present, the expected profit from R&D investment is $\bar{\pi}_F = E\{\pi(x)\} = \int \pi(x)dF(x)$, so a monopolist will invest if $E\{\pi(x)\} - \pi(1) > \rho$. If an NPE is present and both invest, there are two possibilities, $\{x < z\}$ and $\{z < x\}$ where either the firm or else the NPE succeeds in acquiring the patent. As a matter of notation, x will denote the innovation level of a firm, and z the innovation level of the NPE. In what follows, the strategic behavior of both parties is examined — proposition (1) below describes equilibrium investment behavior in the presence of an NPE.

For licensing, there are essentially three options — a fixed fee, a royalty or unit fee, or some combination of fixed fee and royalty. Here, it is assumed that the fee scheme used will be optimal for the rights holder (in terms or overall revenue generation). Typically, this consists of both a fixed fee and unit fee (in the spirit of a two part-tariff). While this seems natural, the determination of the optimal fee structure can be subtle. In any event, this two part structure is used here — further clarification is given in section (4.1) and in the appendix.

If $z < x$ the NPE wins the property right and sets a unit fee of f and can charge a lump sum payment (fixed fee) of $L = \pi(z+f) - \pi(1)$, where $\pi(x+f) = \max_q\{p(q)q - (x+f)q\}$, with solution $q(f)$. This extracts all the surplus, leaving the monopolist indifferent between licensing and not licensing from the NPE. Total license revenue is the sum of lump sum payment and royalty fee, $r(f) = \pi(z+f) - \pi(1) + fq(f)$. Therefore, the license fee f determines overall revenue. With $f > 0$,

$$r'(f) = \frac{\partial \pi}{\partial f} + f\frac{\partial q}{\partial f} + q = -q + f\frac{\partial q}{\partial f} + q = f\frac{\partial q}{\partial f} \leq 0,$$

So that r is maximized at $f = 0$, assuming $\frac{\partial q}{\partial f} \leq 0$. Thus, for the NPE, the optimal (revenue maximizing) licensing scheme is to charge a lump sum fee of $\pi(z) - \pi(1)$.

In the case where both firm and NPE invest, innovation outcomes fall into two regions: $\{(z,x) \mid z < x\}$ where the NPE obtains the property right and $\{(z,x) \mid x \le z\}$ where the firm obtains the right. The probability that the NPE is successful is $\int G(z)dF(z)$ and the probability that the firm is successful is $\int F(x)dG(x)$. In the event that $\{x \le z\}$ the firm obtains $\pi(x)$ and the NPE receives nothing whereas if $\{z < x\}$ then the NPE can extract the full rent from the firm, obtaining $\pi(z) - \pi(1)$ while the firm obtains $\pi(1)$. Thus, the payoff to the firm at (x,z) is $V_F(x,z) = \pi(x) \cdot \mathbf{1}_{\{x<z\}} + \pi(1) \cdot \mathbf{1}_{\{x>z\}}$ where $\mathbf{1}_{\{x<z\}}$ is the indicator function of the event $x < z$. Similarly, the payoff to the NPE at (x,z) is $V_N(x,z) = [\pi(z) - \pi(1)] \cdot \mathbf{1}_{\{x>z\}} + 0 \cdot \mathbf{1}_{\{x<z\}}$. Let $\bar{V}_F = E\{V_F(x,z)\}$ and $\bar{V}_N = E\{V_F(x,z)\}$. When both invest, the net gain for the firm over not investing (given the NPE does) for the firm is $E_F \stackrel{\text{def}}{=} \bar{V}_F - \pi(1)$ with analogous value $E_N \stackrel{\text{def}}{=} \bar{V}_N$ for the NPE. Therefore, both invest if $\rho < \min\{E_F, E_N\}$. (Calculations in the appendix show that $E_F = \int_0^1 [\pi(x) - \pi(1)][1 - G(x)]dF(x)$ and $E_N = \int_0^1 [\pi(z) - \pi(1)][1 - F(z)]dG(z)$.) For ρ between $\min\{E_F, E_N\}$ and $\max\{E_F, E_N\}$, if $E_F < E_N$, the NPE alone invests, and if $E_N < E_F$, the firm alone invests.[1]

If the NPE alone invests, expected licensing revenue is $S_N \stackrel{\text{def}}{=} E\{\pi(z)\} - \pi(1) = \bar{\pi}_G - \pi(1)$, whereas not investing gives a payoff of 0. If the firm alone invests, its expected revenue is $E\{\pi(x)\} = \bar{\pi}_F$ whereas not investing gives a payoff of $\pi(1)$, so the gain to investing is $S_N \stackrel{\text{def}}{=} \bar{\pi}_F - \pi(1)$. Figure 1 depicts the strategic situation.

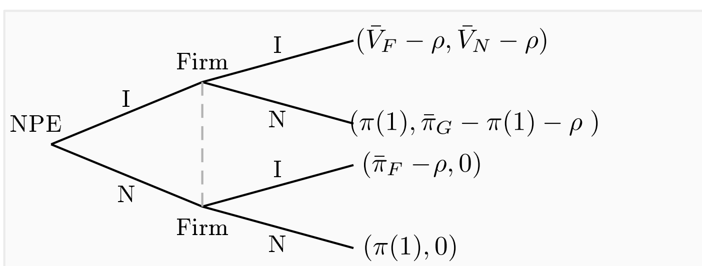


Figure 1: The monopoly case, payoffs (Firm,NPE).

Let $\underline{\rho} = \min\{E_F, E_N\}$ and $\bar{\rho} = \max\{E_F, E_N\}$, $\rho_* = \min\{S_F, S_N\}$ and $\rho^* = \max\{S_F, S_N\}$

Proposition 1: With an NPE, both firm and NPE invest when $\rho < \underline{\rho}$. On the region $[\underline{\rho}, \bar{\rho}]$ only one invests with the firm investing if $E_F > E_N$ and the NPE investing when $E_F < E_N$. On the region $[\bar{\rho}, \rho^*]$

[1] See appendix A for detailed calculations.

only one invests. If $\bar{\rho} < \rho_*$, then on the region $[\bar{\rho}, \rho_*]$ only one invests and it may be either the firm or the NPE. On the region $[\rho_*, \rho^*]$ the entity with the larger value of $\{S_F, S_N\}$ is the sole investor.

Figure (2) illustrates the regions and investment behavior. Equilibrium behavior on the cost region $[0, \bar{\rho}]$ is straightforward: both invest. On the region $[\underline{\rho}, \bar{\rho}]$ the entity with the larger "E" value invests. On the region $[\bar{\rho}, \rho_*]$ there are two equilibria in which one or other entity invests. Finally, on the region $[\rho_*, \rho^*]$ the entity with the larger "S" value invests. In the case, where the distributions G and F are equal, $\underline{\rho} = \bar{\rho}$ and $\rho_* = \rho^*$. If $G \succeq F$ (G first order stochastically dominates F), then $E_N \leq E_F$ and $S_N \leq S_F$ (and $E_N \geq E_F$

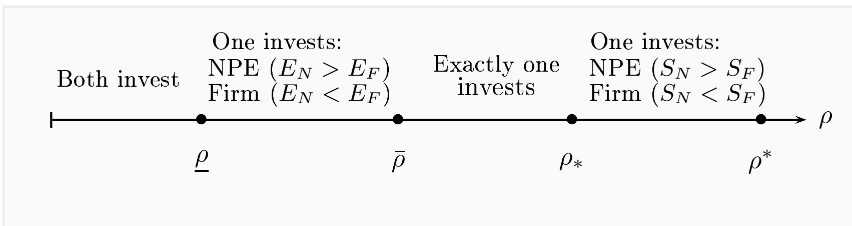


Figure 2: Investment regions.

and $S_N \geq S_F$ if $F \succeq G$). In general, $E_N < E_F$ and $S_N > S_F$ (or $E_N > E_F$ and $S_N < S_F$) are possible orderings (see example (A.1) in appendix A, illustrating this with distributions ordered by second order stochastic dominance). Note that $F \succeq G$ means that F is biased toward drawing a higher technology value and hence is worse distribution of innovation than G.

In general the gap between $\bar{\rho}$ and ρ_* may be large, and in particular even in the case where $F \succeq G$ or $G \succeq F$. For example, suppose that $F \succeq G$ with $E_N > E_F$ and $S_F < S_N$, then $\bar{\rho} = \int(\pi(z) - \pi(1))(1 - F(z))dG(z)$ whereas $\rho_* = \int(\pi(z) - \pi(1))dF(z))$. On the region $[\bar{\rho}, \rho_*]$ just one invests, but it could be either one since both are equilibrium outcomes. However, efficiency requires that the more efficient entity would be the investor.

3.1 Efficiency

For $\rho < \underline{\rho}$, the presence of the NPE unambiguously raises investment and therefore improves the cost distribution. If $S_N > S_F$, then on the region $[\rho_*, \rho^*]$ investment takes place that otherwise would not. The range of costs over which investment takes place is never reduced. However, inefficiency can arise from strategic considerations. To see this, suppose that $G \succeq F$ so that F is the better distribution. On the region $[\bar{\rho}, \rho_*]$ one invests, but it may be either: if it is the NPE, then for costs on this region the presence of the

NPE produces an inferior distribution over the quality of innovation. Note that as presented, both parties make the investment decision simultaneously. If the NPE investment decision is made first and observed by the firm, the unique equilibrium (subgame perfect) has the NPE invest on this region, preempting the firm, even though the firm would draw from a better distribution were it to invest.

4 The Two Firm Case

Turning to the case where there are two firms with demands for firm i and j given by $p^i(q_i, q_j)$ and $p^i(q_i, q_j)$ respectively, and where the demand functions are symmetric in the sense that $p^i(a, b) = p^j(b, a)$. Technologies for i and j are initially at a benchmark level of 1. In this environment firms conduct research resulting in outcomes x_i and x_j, where the lower value, $\min\{x_i, x_j\}$ represents the better technology. Each investing firm draws an innovating technology independently according to a distribution F. Assume that the firm with the better technology, patents the technology so that the competitor must either use the pre-existing technology (at level 1), or license the patented technology. So, if $x_j > x_i$, then j either uses the technology 1 or licenses x_i. Initially, the situation is symmetric with both having technology 1; afterwards, once investments are made and technology drawn, the situation involves technology licensing, where the licensor is the party with the better technology. To begin, section (4.1) considers the optimal licensing decision from the perspective of a firm and that of the NPE. In section (4.2) strategic behavior in the absence of an NPE is considered. This provides a benchmark for the subsequent discussion of strategy behavior when an NPE is active (section (4.3)).

4.1 Licensing

Models of licensing are discussed in [7, 8, 9, 12], among others. These describe a wide range of licensing models, including procedures based on Nash bargaining, alternating offers bargaining, patent valuation using the Shapley value, auction methods, fixed fees, royalty fees and other procedures. Here, the licensing scheme adopted is essentially a two-part tariff consisting of a royalty fee and a fixed fee. In what follows, the structure of this scheme is examined in some detail to provide the framework for the study of strategic behavior of the firms and NPE. (Remark (4.1) summarizes the main observations. Lemmas (1), (2) and (3) in appendix B provide additional details.)

Consider first the environment without the NPE and in particular the case where $x_i < x_j$ so that firm i has the better technology and j must license or use the pre-existing technology. Assume for the moment that there is no active NPE so that licensing between firms is the only issue. Let $\pi^j(c_i, c_j)$ be the profit to firm i and j at the equilibrium output choices when costs are (c_i, c_j). Initially, let $(c_i, c_j) = (1, 1)$ and after innovation realization, with i the patent holder, $(c_i, c_j) = (x_i, x_i + f)$, assuming j licenses from i, while without licensing the cost structure is $(c_i, c_j) = (x_i, 1)$. [2] With licensing fees, operating profits are:

$$\pi^i(x_i, x_i + f; q_i, q_j) = p^i(q_i, q_j)q_i - x_i q_i, \qquad \pi^j(x_i, x_i + f; q_i, q_j) = p^j(q_i, q_j)q_j - (x_i + f)q_j$$

In addition, firm i receives licensing revenue from firm j. This consists of fq_j and possibly a fixed fee, F. Let $(\hat{q}_i(f, x_i), \hat{q}_j(f, x_i))$ be the equilibrium outputs of i and j respectively when i sets the royalty fee at f (so that i operates with cost x_i and j operates with cost $x_i + f$). Let $\pi^i(x_i, x_i + f)$ and $\pi^j(x_i, x_i + f)$ be equilibrium profits in this case.

With x_i given and royalty fee f, revenue to i from royalties from j is $f\hat{q}_j(f, x_i)$. The highest possible royalty fee that i can charge is $f = 1 - x_i$ for the use of technology x_i (which eliminates the benefit to j from licensing). Profit of j after payment of royalties is $\pi^j(x_i, x_i + f) = p^j\hat{q}_j - (x_i + f)\hat{q}_j$; in the absence of licensing, firm j obtains profit $\pi^j(x_i, 1)$. Therefore the maximum fixed fee that j is willing to pay in addition to the royalty $(f\hat{q}_j)$ is $F = \pi^j(x_i, x_i + f) - \pi^j(x_i, 1)$. So, the maximum revenue i can obtain from licensing to j is:

$$R(x_i, f) \quad = \quad [\pi^j(x_i, x_i + f) - \pi^j(x_i, 1)] + f\hat{q}_j(f, x_i) \tag{1}$$

Firm i earns profit in addition to the licensing revenue so that i's profit plus licensing revenue is:

$$\hat{\pi}^i(x_i, f) = \pi^i(x_i, x_i + f) + R(x_i, f) \tag{2}$$

Let $\psi(x_i)$ be the maximal revenue from profit plus license fees obtained by i with licensing to firm j:

$$\begin{aligned}
\psi(x_i) &= \max_{f \le 1 - x_i} \hat{\pi}^i(x_i, f) = \hat{\pi}^i(x_i, \hat{f}(x_i)) = \max_{f \le 1 - x_i} \{\pi^i(x_i, x_i + f) + [\pi^j(x_i, x_i + f) - \pi^j(x_i, 1)] + f\hat{q}_j(f, x_i)\} && (3) \\
&= \max_{f \le 1 - x_i} \left\{[p^i\hat{q}_i - x_i\hat{q}_i] + [p^j\hat{q}_j - x_i\hat{q}_j]\right\} - \pi^j(x_i, 1) && (4) \\
&= \psi^*(x_i) - \pi^j(x_i, 1), \quad \text{where } \psi^*(x_i) \stackrel{\text{def}}{=} \max_{f \le 1 - x_i} \left\{[p^i\hat{q}_i - x_i\hat{q}_i] + [p^j\hat{q}_j - x_i\hat{q}_j]\right\} && (5)
\end{aligned}$$

[2] Here, it is assumed that the best firm gets the patent. There e are many aspects to the strategic management of IP and patent acquisition (see [1, 2, 6] for some perspectives). For example, if technology improvement proceeds incrementally with the firm that draws the better technology developing the technology faster incrementally and patenting as the firm goes, the winner is always ahead and owns all the improvement from 1 down to x_i, if i is the winner. An alternative assumption here would be that each firm retains the right to use what technology it has developed, even if one firm has a patent on superior technology.

From this expression, the optimal licensing scheme involves maximizing total profit in the two markets, subject to $0 \leq f \leq 1 - x_i$. Note that the maximization is of joint profit through the impact of f on $(\hat{q}_i, \hat{q}_j)$. Denote the optimal choice of f, $\hat{f}$ so that $\hat{f} = \hat{f}(x_i) \stackrel{\text{def}}{=} \arg\max_{\{f \leq 1-x_i\}} \hat{\pi}^i(x_i, f)$. The corresponding licensing revenue is $\hat{R} = \hat{R}(x_i) \stackrel{\text{def}}{=} R(\hat{f}(x_i); x_i)$ and $\psi(x_i) = \hat{\pi}^i(x_i, \hat{f})$.

When an NPE is present and licenses technology z, the unit fee of f gives both firms a unit cost of $z + f$ and equilibrium output $\tilde{q}_i(f; z) = \tilde{q}_j(f; z)$. The corresponding profit for i is

$$\tilde{\pi}^i(z, f) = \pi^i(z + f, z + f) = p^i \tilde{q}_i - (z + f)\tilde{q}_i$$

The total revenue for the NPE, using a royalty and (maximum) fixed fee is:

$$N(z, f) = [\tilde{\pi}^i(z, f) - \pi_i(1, 1)] + f\tilde{q}_i + [\tilde{\pi}^j(z, f) - \pi_j(1, 1)] + f\tilde{q}_j \tag{6}$$

(Here, it is assumed that the NPE treat firms symmetrically so that full rent extraction puts each firm at the pre-innovation profit.) Maximizing this gives maximal total revenue:

$$\begin{aligned}
\varphi(z) &= \max_{0 \leq f \leq 1-z} N(z, f) = \max_{0 \leq f \leq 1-z} \left\{[\tilde{\pi}^i(z, f) - \pi_i(1, 1)] + f\tilde{q}_i + [\tilde{\pi}^j(z, f) - \pi_j(1, 1)] + f\tilde{q}_j\right\} &(7)\\
&= \max_{0 \leq f \leq 1-z} \left\{[p^i \tilde{q}_i - z\tilde{q}_i] + [p^j \tilde{q}_i - z\tilde{q}_j]\right\} - \pi^i(1, 1) - \pi^j(1, 1) &(8)\\
&= \varphi^*(z) - \pi^i(1, 1) - \pi^j(1, 1), \quad \text{where } \varphi^*(z) \stackrel{\text{def}}{=} \max_{0 \leq f \leq 1-z} \left\{[p^i \tilde{q}_i - z\tilde{q}_i] + [p^j \tilde{q}_i - z\tilde{q}_j]\right\} &(9)
\end{aligned}$$

Let the solution value of f be $\tilde{f} = \arg\max_{\{0 \leq f \leq 1-z\}} N(z, f)$. Note that $\tilde{f}$ is chosen to maximize total profit through the impact of f on $(\tilde{q}_i, \tilde{q}_j)$. Observe that $\psi^*(1) = \varphi^*(1) = \pi^i(1, 1) + \pi^j(1, 1)$, so that $\psi(1) = \pi^i(1, 1)$ and $\varphi(1) = 0$. The properties of optimal licenses are discussed in appendix B.

REMARK 4.1: Under general conditions, $R(x_i, f)$ is maximized at $f = 0$. However, while $R(f; x_i)$ is generally decreasing in f, $\pi^i(x_i, x_i + f)$ is increasing in f and for firm i the optimal f maximizes $\hat{\pi}^i(x_i, f)$ which is generally increasing in f at $f = 0$ (because raising f contracts j's output implicitly raising i's direct profit). Therefore, in general the optimal licence fee is positive and the optimal licensing structure is a two part tariff consisting of a fixed unit royalty fee, $f > 0$, and a lump sum fee. In the case of the NPE, a positive license fee, $\tilde{f}$ raises profit by curtailing output and this profit can be accessed by the NPE through the fixed fee.[3]

REMARK 4.2: An alternative licensing policy for the firm is to establish a licensing unit — which then licenses to both its own operating unit and to the other firm. In this case, the firm's licensing unit acts

[3] See appendix B, lemmas 1, 2 and 3.

as an NPE extracting as much license revenue as possible both from the other firm and from its own production unit. Thus, if firm i obtains the patent with technology x_i, revenue from the licensing unit is $\varphi(x_i)$ and (residual) revenue to the firm is $\pi^i(1,1)$ for total revenue of $\varphi(x_i) + \pi^i(1,1)$. In the linear case, $\psi^*(x_i) - \pi^j(x_i,1) \geq \varphi(x_i) + \pi^i(1,1)$ (see proposition 7), so that setting up an NPE is less profitable in that case.

4.2 Equilibrium Investment and Licensing without an NPE.

In terms of investment decisions, each firm may or may not invest, so there are three possibilities: both firms invest, one invests, or neither invest. If one or both invest, then subsequently one will become the patent holder and the other the potential licensee. If neither invests, then the status quo remains. These possibilities are considered in turn. In what follows, the expected benefit to a firm in each possible scenario is determined (the values A, B, C and D below.) From these, strategic investment choice is seen from the matrix (13) and the equilibrium number of investors as a function of the investment cost is shown in figure (4).

If both firms invest, depending on the realizations of (x_i, x_j) either $x_i < x_j$ or $x_j < x_i$, ignoring ties. The outcomes are symmetric, so consider the case where $x_i < x_j$ so that i becomes the patent holder. In this case, with licensing, the payoff to i is $\psi(x_i) = \hat{\pi}^i(x_i, \hat{f})$ and to j, $\pi^j(x_i, 1)$. The payoff pairs are depicted in figure (3). Thus, the expected payoff to either firm is:

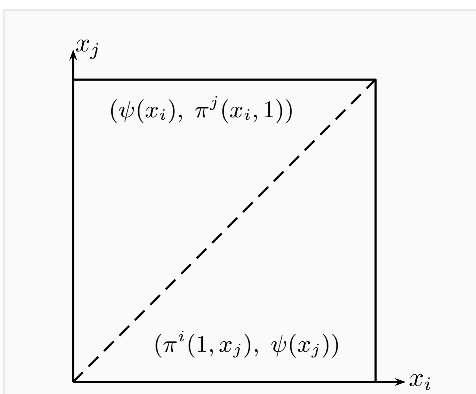


Figure 3: Total Revenue

$$A = \int_0^1 \int_{x_i}^1 \psi(x_i) dF(x_j) dF(x_i) + \int_0^1 \int_0^{x_i} \pi^i(1, x_j) dF(x_j) dF(x_i) \tag{10}$$

If just one firm invests, say i, i obtains the patent at any profile $(x_i, 1)$ so the payoff pair is $(\hat{\pi}^i(x_i, \hat{f}),\ \pi^j(x_i, 1))$. The expected payoff to the investing firm, i, is:

$$B = \int_0^1 \int_0^1 \psi(x_i) dF(x_j) dF(x_i) \tag{11}$$

and to the non-investing firm:

$$C = \int_0^1 \int_0^1 \pi^i(1, x_j) dF(x_j) dF(x_i) = \int_0^1 \int_0^1 \pi^j(x_i, 1) dF(x_i) dF(x_j) \tag{12}$$

Finally, when neither invest the payoff is $D = \pi^i(1,1) = \pi^j(1,1)$. (Subsequently, for convenience, the notation $dF_j = dF(x_j)$ and $dF_i = dF(x_i)$ may sometimes be used.) This defines the matrix game where each firm may invest (I) or not invest (N):

$$\begin{array}{cc} & \begin{array}{cc} I & N \end{array} \\ \begin{array}{c} I \\ N \end{array} & \begin{pmatrix} (A, A) & (B, C) \\ (C, B) & (D, D) \end{pmatrix} \end{array} \tag{13}$$

(From the definitions, $B > A > D > C$ and proposition (2) shows that $A - C < B - D$.) With investment cost ρ, proposition (2) describes the possible equilibria. Figure (4) depicts the investment pattern in terms of ρ. From a strategic perspective, $A - C$ measures the incentive to invest when the other firm does, and $B - C$ measures that incentive when the other firm does not invest. Proposition 2 characterizes the equilibria (See appendix C for proofs).

Proposition 2: Payoffs satisfy $0 \leq A - C \leq B - D$. With investment cost ρ, if $\rho < A - C$ there is a unique equilibrium where both firms invest. If $A - C < \rho < B - D$, then there are two equilibria where one or other firm invests. And if $B - D < \rho$ then the unique equilibrium has neither firm invest.

For low investment cost, $\rho < A - C$, (I, I) is the unique equilibrium (investing strictly dominates not investing.) As investment cost rises to intermediate levels, both (I, N) and (N, I) are equilibria. For $\rho > B - D$, N is a dominant strategy. The number of investing firms as a function of ρ is depicted in Figure (4).

The next section examines how this investment pattern is affected by the presence of an NPE.

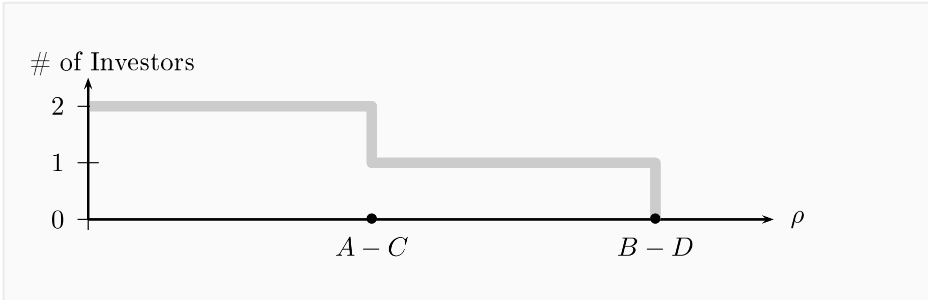


Figure 4: Equilibrium Investment

4.3 Investment and Licensing with an NPE.

The presence of an NPE alters the strategic situation for the firms in that both the expected benefits to investing may be less and the expected cost to not investing greater. Just as the firms must make a choice regarding investment, so must the NPE. Here, it is assumed that the firms know whether the NPE is engaged in R&D or not, so that if parties invest, they are competing with the NPE to win the patent: the firms are aware of the NPE's investment strategy, but not the outcome of its R&D. The framework here is easily adapted to allow for the case where the firms observe the success of the NPE before investing, but in that case must decide whether or not to try and beat the NPE with a better discovery. As in section (4.2) it's necessary to determine the payoffs for each investment choice of the firms and NPE (sections (4.3.1) and (4.3.2)), and from there proceed to the characterization of equilibrium (section 4.4).

4.3.1 The firm's payoff in the presence of an NPE

When a NPE is present, conditional on a realization of z, payoff flows to the two firms are given in figure (5). When both firms invest, the expected payoff is (with notation $dF_j = dF(x_j)$):

$$A(z) = \int_0^z \int_{x_i}^1 \psi(x_i) dF_j dF_i + \int_0^z \int_{x_i}^1 \pi^j(x_i, 1) dF_j dF_i + \int_z^1 \int_z^1 \pi^i(1,1) dF_j dF_i \tag{14}$$

When one invests, the investing firm obtains:

$$B(z) = \int_0^z \int_0^1 \psi(x_i) dF_j dF_i + \int_z^1 \int_0^1 \pi^i(1,1) dF_j dF_i \tag{15}$$

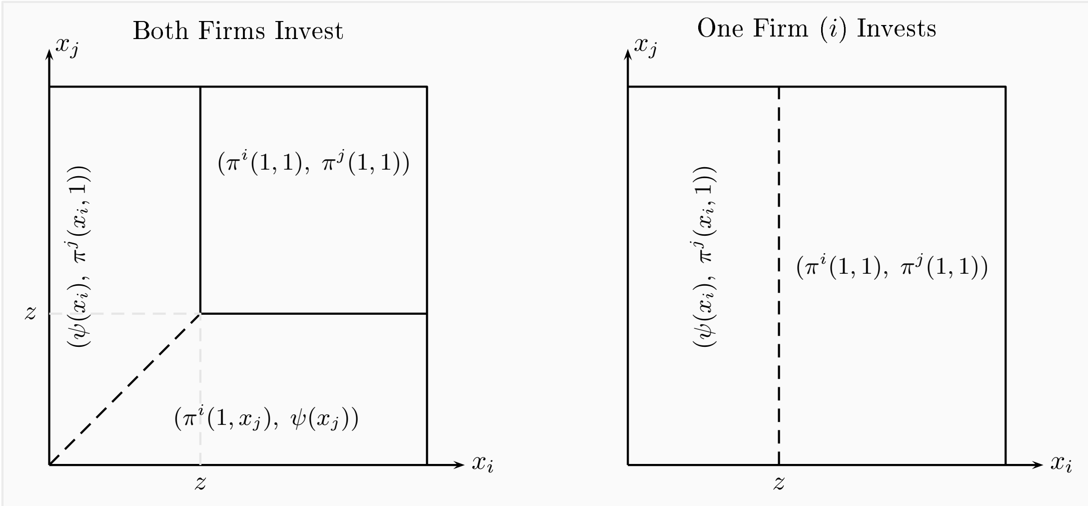


Figure 5: Total Revenue

And the non-investing firm gets:

$$C(z) = \int_0^z \int_0^1 \pi^j(x_i, 1) dF_j dF_i + \int_z^1 \int_0^1 \pi^i(1,1) dF_j dF_i \tag{16}$$

Finally, let $D(z) = \pi^i(1,1)$. The corresponding game is:

$$\begin{array}{cc} & \begin{array}{cc} I & N \end{array} \\ \begin{array}{c} I \\ N \end{array} & \begin{pmatrix} (A(z), A(z)) & (B(z), C(z)) \\ (C(z), B(z)) & (D(z), D(z)) \end{pmatrix} \end{array}$$

The matrix gives the firms payoffs, given their choices and given the realization of the NPE's innovation. The discussion will focus on the case where the NPE's innovation is not observed prior to the firm's investment decisions, so it is the averages of these functions that are relevant for the firms choices. Subsequently, averages such as that of $A(z)$ will be given as: $\bar{A} = \int A(z) dG(z)$.

Proposition 3: The differences $A(z) - C(z)$ and $B(z) - D(z)$ satisfy the following properties:

1. $A(0) - C(0) = B(0) - D(0) = 0$, $A - C = A(1) - C(1) \leq B(1) - D(1) = B - D$,

2. $0 \leq A'(z) - C'(z) \leq B'(z) - D'(z)$.

It follows from 1 and 2 that $A(z) - C(z) \leq B(z) - D(z)$ for all z.

From proposition (3), $0 \leq \bar{A} - \bar{C} \leq \bar{B} - \bar{D}$, $\bar{A} - \bar{C} \leq A - C$, and $\bar{B} - \bar{D} \leq B - D$. In particular, these inequalities yield proposition (2).

4.3.2 The NPE's payoff

Recall that $\varphi(z)$ (see equation (7)) gives the total revenue for the NPE when licensing to both firms. The probability of success of the NPE, in terms of being able to license depends on the number of firms investing. If both invest, given z, the return $\varphi(z)$ is obtained in the region where $z < \min\{x_i, x_j\}$, and so the expected payoff, given z is $\int_z^1 \int_z^1 \varphi(z) dF_j dF_i = \varphi(z)[1 - F(z)]^2$. If just one firm invests, say i, then the NPE licenses on the region $\{z < x_i\}$ so the expected payoff to the NPE given z is $N_1(z) = \int_z^1 \int_0^1 \varphi(z) dF_j dF_i = \varphi(z)[1 - F(z)]$. Finally, if neither firm invests, the NPE licenses for sure and the expected payoff is $N_0(z) = \varphi(z)$. Let

$$N_k(z) = \varphi(z)[1 - F(z)]^k, \quad k = 0, 1, 2$$

denote the expected revenue of the NPE given innovation z, given k firms invest. From the definitions, $N_2(z) \leq N_1(z) \leq N_0(z)$. The expected revenue of the NPE from investment when k firms invest is:

$$\bar{N}_k = \int_0^1 \varphi(z)[1 - F(z)]^k dG(z), \quad k = 0, 1, 2$$

Recall that

$$\varphi(z) \quad = \quad \varphi^*(z) - \pi^i(1,1) - \pi^j(1,1), \quad \text{where } \varphi^*(z) \stackrel{\text{def}}{=} \max_{0 \leq f \leq 1-z} \left\{[p^i \tilde{q}_i - z\tilde{q}_i] + [p^j \tilde{q}_i - z\tilde{q}_j]\right\} \qquad (17)$$

At $z = 1$, the only possible choice of f is $f = 0$ and then $p^\alpha \tilde{q}_\alpha - z\tilde{q}_\alpha = \pi^\alpha(1,1)$, $\alpha = i, j$. In this case $\varphi(z) = 0$. When $z = 0$, $\varphi^*(0) \geq \pi^i(0,0) + \pi^j(0,0)$, so $\varphi(0) \geq 2[\pi^i(0,0) - \pi^i(1,1)]$ (using symmetry). Note also that $\varphi^*(z)$ is decreasing in z. It is useful to plot the functions that determine the strategic calculations of both the firms and the NPE.

4.4 Equilibrium with an NPE

Although the model is quite simple, the structure of the equilibrium set is somewhat complex because there are a large number of possible equilibrium outcomes, depending on the exact shape of the profit function and innovation distributions. In terms of the benefit from investment, there are two aspects to bear in mind — the volume of investment and the performance of the investment (because the firms and NPE have different

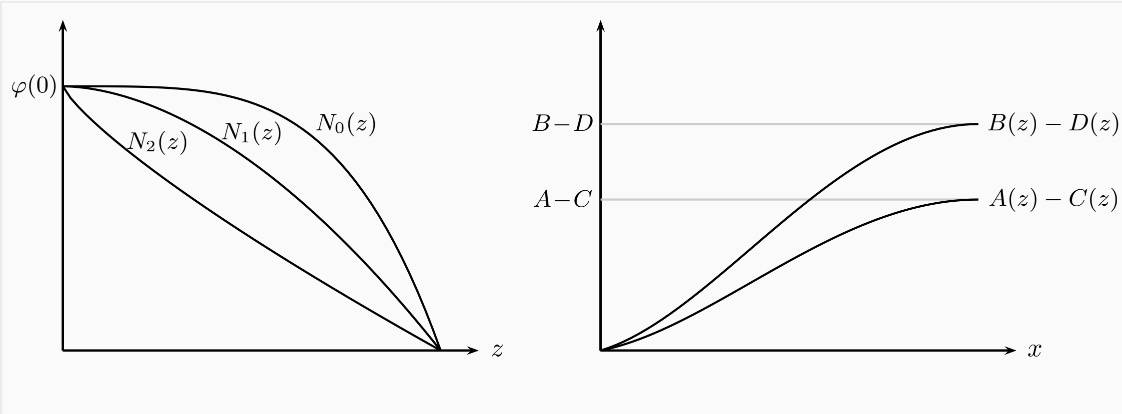


Figure 6: The functions $N_k(z)$, $A(z) - C(z)$ and $B(z) - D(z)$.

success performance, as measured by F and G). So, for example, from a welfare perspective, the case where the NPE alone invests may be different from that where a single firm invests, even thought the investment expenditure is the same. These issues are considered first for the general case and then for the linear model (section (5)) where the comparisons are simpler.

Proposition (4) describes equilibrium behavior in the NPE environment. Although the exit thresholds are determined by $\bar{A} - \bar{C}$ and $\bar{B} - \bar{D}$ for the firms and $\bar{N}_2$, $\bar{N}_1$, $\bar{N}_0$ for the NPE, the thresholds are not *fully* determined by these: the characterization also requires parameters from the non-NPE environment ($A - C$ and $B - D$). The reason is simple. If, for example, the NPE chooses not to invest, regardless of the behavior of the firms, then the firms are effectively in a non-NPE environment with an analogous strategic situation.

Proposition 4: Equilibrium investment as ρ increases is as follows:

1. For $\rho \in [0, \bar{A} - \bar{C}]$:
   (a) If $\rho \leq \min\{\bar{N}_2, \bar{A} - \bar{C}\}$ the investment structure is ffn.
   (b) If $\min\{\bar{N}_2, \bar{A} - \bar{C}\} < \rho \leq \bar{A} - \bar{C}$ the investment structure is ff.
2. For $\rho \in (\bar{A} - \bar{C}, \bar{B} - \bar{D}]$:
   (a) If $\bar{A} - \bar{C} < \rho \leq \min\{\bar{N}_1, \bar{B} - \bar{D}\}$ the investment structure is fn.
   (b) If $\min\{\bar{N}_1, \bar{B} - \bar{D}\} < \rho \leq \min\{\max\{\bar{N}_1, A - C\}, \bar{B} - \bar{D}\}$ the investment structure is ff.
   (c) If $\max\{\bar{N}_1, A - C\} < \rho \leq \bar{B} - \bar{D}$, then investment structure is f.
3. For $\rho \in (\bar{B} - \bar{D}, B - D]$:

(a) If $\bar{B}-\bar{D}<\rho\leq\max\{\bar{B}-\bar{D},\bar{N}_0\}$ the investment structure is n.

(b) If $\max\{\bar{B}-\bar{D},\bar{N}_0\}<\rho\leq\max\{\max\{\bar{B}-\bar{D},\bar{N}_0\},A-C\}$ the investment structure is ff.

(c) If $\max\{\max\{\bar{B}-\bar{D},\bar{N}_0\},A-C\}<\rho\leq\max\{\max\{\bar{B}-\bar{D},\bar{N}_0\},B-D\}$ the investment structure is f.

(d) If $\max\{\max\{\bar{B}-\bar{D},\bar{N}_0\},B-D\}<\rho$ there is no investment.

4. For $\rho\in(B-D,\infty)$:

- If $\rho>\max\{\bar{N}_0,B-D\}$, or there is no investment.
- If $\bar{N}_0>B-D$ the NPE alone invests for $\rho\in(B-D,\bar{N}_0)$.

To motivate the description of equilibrium investment in proposition (4), consider a few cases.

Remark 4.3: In all figures throughout the paper, broad lines, ▬▬, indicate investment levels in the non-NPE environment with two firms and dashed lines, ‐ ‐ ‐ ‐, indicate investment in the NPE environment. The notation *fn* in figure (7) means that one firm (f) and the NPE (n) invest; similarly, f means that just one firm alone invests, and so on. In the non-NPE environment, only firms can invest so there is no room for ambiguity.

Consider the case where $\bar{A}-\bar{C}<\bar{N}_1<A-C<\bar{B}-\bar{D}$ or where $\bar{A}-\bar{C}<A-C<\bar{N}_1<\bar{B}-\bar{D}$. In the non-NPE environment, the two firms will invest for ρ up to $A-C$ and one will invest for $A-C<\rho\leq\bar{B}-\bar{D}$ (since $\bar{B}-\bar{D}\leq B-D$). Next, consider the NPE environment. For $\rho>\bar{A}-\bar{C}$, there is no equilibrium where

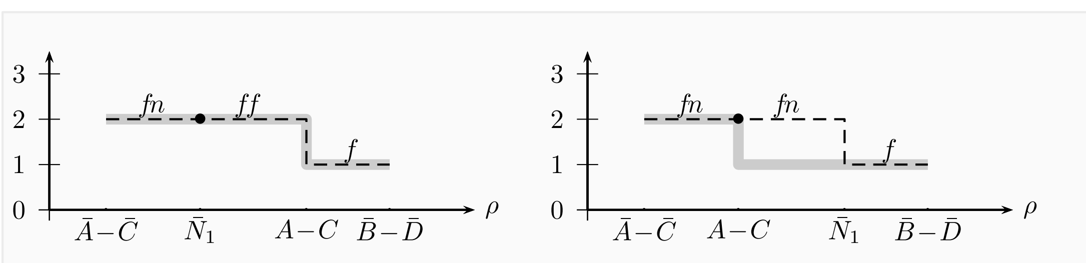


Figure 7: Investment on $(\bar{A}-\bar{C},\bar{B}-\bar{D}]$.

both firms invest if the NPE does also. If the NPE invests, the expected gross return when one firm invests is $\bar{N}_1$ and investment cost is ρ — so provided $\bar{A}-\bar{C}<\rho\leq\bar{N}_1<\bar{B}-\bar{D}$, the NPE will invest as will one firm. This is indicated in the figure by *fn*. When $\bar{N}_1<\rho\leq\bar{B}-\bar{D}$ even if the NPE invests, at least one firm will also, and in this case the NPE will not invest since $\rho > \bar{N}_1$. But, in this case, on the interval $\rho \in (\bar{N}_1, A{-}C]$ both firms will invest (indicated by ff). Finally, when $\rho \in (A{-}C < \bar{B}{-}\bar{D}]$, again the NPE will no invest, and because $\rho > A{-}C$, just one firm will invest. (This is indicated by f). Similarly, considering the case $\bar{A}{-}\bar{C} < A{-}C < \bar{N}_1 < \bar{B}{-}\bar{D}$ suppose that $\rho \in (A{-}C, \bar{N}_1]$. In this case, the NPE will invest as will one firm — since the firm will invest (given the presence of the NPE), for ρ up to $\bar{B} - \bar{D}$, so that the NPE and one firm invest (nf). And so on.

Let $I_N(\rho)$ and $I_0(\rho)$ be total investment when the NPE is present ($I_N(\rho)$), and when the NPE is not present ($I_0(\rho)$). The following proposition characterizes the levels of investment in the two regimes. From proposition (4) the level of investment may be characterized as a function of investment cost ρ.

PROPOSITION 5: The impact of the NPE on investment (relative to the environment with just two firms) is as follows:

1. For $\rho \in [0, \bar{A} - \bar{C}]$, $I_N(\rho) \geq I_0(\rho)$, with strict inequality for $\rho \leq \bar{N}_2$.

2. For $\rho \in (\bar{A}{-}\bar{C}, \bar{B}{-}\bar{D}]$, $I_N(\rho) \geq I_0(\rho)$, with equality if $A{-}C \geq \min\{\bar{B}{-}\bar{D}, \bar{N}_1\}$. If $A{-}C < \min\{\bar{B}-\bar{D}, \bar{N}_1\}$ then $I_N(\rho) > I_0(\rho)$ for $\rho \in (A{-}C, \min\{\bar{B} - \bar{D}, \bar{N}_1\}]$. If $A{-}C < \min\{\bar{B} - \bar{D}, \bar{N}_1\} < \bar{B} - \bar{D}$, then $I_N(\rho) = I_0(\rho)$ for $(\bar{N}_1, \bar{B} - \bar{D}]$.

3. For $\rho \in (\bar{B} - \bar{D}, B - D]$, $I_N(\rho) \leq I_0(\rho)$, with equality if $\min\{A{-}C, \bar{N}_1\} \leq \bar{B} - \bar{D}$. If $\min\{A-C, \bar{N}_1\} > \bar{B} - \bar{D}$ then $I_N(\rho) < I_0(\rho)$ for $\rho \in (\bar{B} - \bar{D}, \min\{A{-}C, \bar{N}_1\}]$ and $I_N(\rho) = I_0(\rho)$ for $\rho \in (\min\{A{-}C, \bar{N}_1\}, B{-}D]$.

4. For $\rho \in (B - D, \infty)$, if $\bar{N}_0 \leq B{-}D$, then $I_N(\rho) = I_0(\rho) = 0$ and if $\bar{N}_0 > B - D$, $I_N(\rho) > I_0(\rho)$ for $\rho \in (B{-}D, \bar{N}_0]$ and $I_N(\rho) = I_0(\rho) = 0$ for $\rho \in (\bar{N}_0, \infty)$.

Although this appears complicated, the main insights can be read from figure (7) (figure (11) in appendix C) and figure (11). For example, if $\bar{A}{-}\bar{C} < \bar{N}_1 < A{-}C < \bar{B}{-}\bar{D}$, then on $(\bar{A}{-}\bar{C}, \bar{N}_1]$ one firm and the NPE invest, whereas on $(\bar{N}_1, A{-}C]$ both firms invest and the NPE does not, while on $(A - C, \bar{B} - \bar{D}]$ just one firm invests. By contrast, in the environment without the NPE, two firms invest for all ρ in the region $(\bar{A}{-}\bar{C}, A{-}C]$ and one invests on $(A{-}C, \bar{B}{-}\bar{D}]$. In each case, there are two investors, but the distribution over outcome will be different because on $(\bar{A}-\bar{C}, \bar{N}_1]$ in one case a firm and the NPE invest, while in the other two firms invest.[4] And, if $A{-}C < \bar{B}{-}\bar{D}$ (again considering the NPE case), from figure (11), if $\bar{N}_0 < \bar{B} - \bar{D}$

[4]Alternatively, if $\bar{A}-\bar{C} < A-C < \bar{N}_1 < \bar{B}-\bar{D}$, in the non-NPE environment just one firm invests for ρ in the interval

one firm invests on the entire interval $(\bar{B}-\bar{D}, B-D]$, whereas if $\bar{N}_0 \in (\bar{B}-\bar{D}, B-D]$ the NPE alone invests for $\rho \in (\bar{B}-\bar{D}, \bar{N}_0]$ and a single firm invests for $\rho \in (\bar{N}_0, B-D]$, contrasting with the non-NPE case where a single firm invests on the entire interval. Of course, in this model, the efficacy of investment, comparing the NPE and the firms, varies because the outcome distributions are different (F and G).

5 The Linear Case

It turns out that in the linear model, the various thresholds are unambiguously ordered. This is discussed next. In the linear model, price is $P(Q) = a - bQ$, the cost function $c_i(q_i) = c_i q_i$ for firm i and the technology distribution is uniform, $F(y) = G(y) = y$.

Proposition 6: In the linear case:

$$\psi^*(x_i) \quad = \quad \frac{1}{9b}[2(a-x_i)^2 + (a-1)(1-x_i)] \quad \text{and} \quad \varphi^*(z) = \begin{cases} \frac{1}{4b}(a-z)^2, & z \le \frac{4-a}{3} \\ \frac{2}{9b}[a-3z+2][a-1], & z > \frac{4-a}{3} \end{cases}$$

From these expressions:

$$\varphi^*(y) - \psi^*(y) = \begin{matrix} \frac{1}{36b}(a+y-2)^2, & y \le \frac{4-a}{3} \\ \frac{1}{9b}(1-y)(2y+a-3), & y \ge \frac{4-a}{3} \end{matrix}$$

so that $\varphi^*(y) - \psi^*(y) \ge 0$ for all y.[5]

It may be worth noting that for a firm. licensing directly (rather than establishing an NPE to administer the IP and collect fees) is preferable. To see this, recall that $\psi(x) = \psi^*(x) - \pi^j(x,1)$ and $\varphi(x) = \varphi^*(x) - \pi^i(1,1) - \pi^j(1,1)$. If the firm establishes a NPE for licensing, the license revenue is $\varphi(x)$ and in addition the firm operation yields $\pi^i(1,1)$ after payment of all license fees. Therefore the two alternative revenues are: $\psi(x) = \psi^*(x) - \pi^j(x,1)$ and $\varphi(x) + \pi^i(1,1) = \varphi^*(x) - \pi^j(1,1)$. For the firm, establishing an NPE to manage intellectual property is preferable if and only if $\psi(x) \ge \varphi(x) + \pi^i(1,1)$. Or equivalently, if $\varphi^*(x) - \psi^*(x) \ge \pi^j(1,1) - \pi^j(x,1)$. Proposition (7) confirms that this condition is satisfied.

$(A-C, \bar{N}_1]$ whereas one firm and the NPE invest in the environment with the NPE. In this case, the presence of the NPE unequivocally improves the distribution over outcomes.

[5] Calculations are in appendix D.

PROPOSITION 7: For all x,

$$\varphi^*(x) - \psi^*(x) \leq \pi^j(1,1) - \pi^j(x,1)$$

Consequently, direct licensing by the firm is preferable.

5.1 Payoff parameters.

For a firm, the key parameters affecting behavior are:

$A - C$	$B - D$	$\bar{A} - \bar{C}$	$\bar{B} - \bar{D}$
$\frac{1}{9b}[-\frac{5}{2} + \frac{8}{3}a]$	$\frac{1}{9b}[-\frac{19}{6} + \frac{7}{2}a]$	$\frac{1}{9b}[-\frac{53}{30} + \frac{23}{12}a]$	$\frac{1}{9b}[-\frac{25}{12} + \frac{7}{3}a]$

For the NPE, depending on the number of firms investing, the expected payoff is:

$$\bar{N}_0 = \begin{cases} \frac{1}{9b}[-\frac{11}{9} + \frac{5}{3}a + \frac{1}{3}a^2 - \frac{1}{36}a^3], & a < 4 \\ \frac{1}{9b}[3a - 3], & a \geq 4 \end{cases}$$

$$\bar{N}_1 = \begin{cases} \frac{1}{9b}[-\frac{22}{27} + \frac{34}{27}a + \frac{1}{9}a^2 + \frac{1}{108}a^3 - \frac{1}{432}a^4], & a < 4 \\ \frac{1}{9b}[2a - 2], & a \geq 4 \end{cases}$$

$$\bar{N}_2 = \begin{cases} \frac{1}{9b}[-\frac{479}{810} + \frac{155}{162}a + \frac{7}{81}a^2 - \frac{1}{324}a^3 + \frac{1}{648}a^4 - \frac{1}{3240}a^5], & a < 4 \\ \frac{1}{9b}[\frac{3}{2}a - \frac{3}{2}], & a \geq 4 \end{cases}$$

Comparison of these gives, for all $a \geq 2$ (take $a \geq 2$ so that profit is always non-negative):

$$\bar{N}_2 < \bar{A} - \bar{C} < \bar{N}_1 < \bar{B} - \bar{D} < A - C < \bar{N}_0 < B - D$$

Figure (13) in the appendix provides a plot as a varies. Given these parameter relations, one may graph the investment pattern as cost varies. In figure (8) the investment patterns with the NPE and in its absence are shown. (The notation ff indicates that (only) the two firms invest; whereas fn indicates that one firm and the NPE invest.) As the figure illustrates, the level of investment is higher with the NPE for low levels of cost ($\rho < \bar{N}_2$). Above $\bar{N}_2$, investment is always as high in the absence of the NPE, and sometimes higher (on the region $[\bar{B}-\bar{D}, A-C]$).

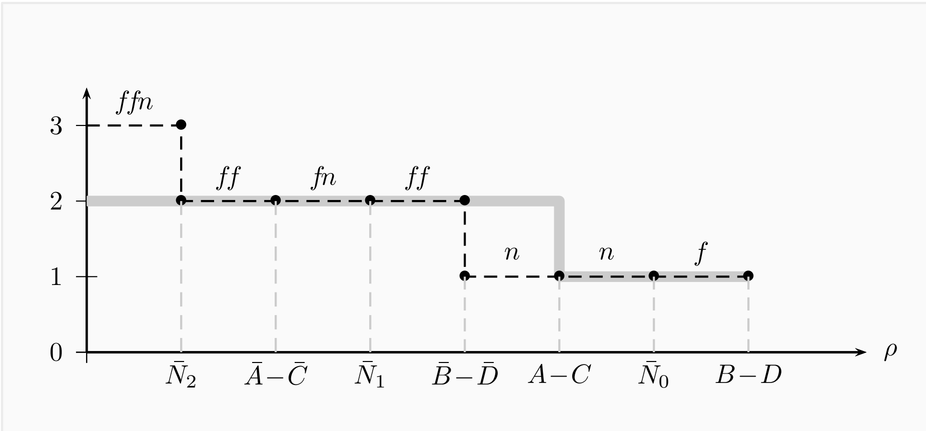


Figure 8: Investment.

Remark 5.1: Because the thresholds vary continuously with the parameters, if the distribution of the NPE were poorer than that of a firm, then on the interval $[\bar{A}-\bar{C}, \bar{N}_1]$ the impact of the NPE is to lower the overall quality of the technology draw relative to the case where there is no NPE.

5.2 The case with three firms

The previous discussion in sections (4.2), (4.3) and (4.4) considers the impact of the presence on an NPE in a two firm two market environment. How would investment be impacted by the addition of third firm (vis-a-vis the addition of the NPE)? The impact on overall investment resulting from the addition of a third firm and a third market contrasts with the addition of an NPE to an environment with two differentiated markets, where the presence of the NPE leaves the market structure unchanged. One simple case to consider is where the two firms supply the same market, so that a third firm is just an additional supplier to this market. Of course, adding third firm changes the degree of competition (unlike the addition of an NPE), so that comparing the overall level of investment in these two cases may be questionable since the market structure is altered. This concern aside, one may compare the impact on investment of the presence of the NPE vis-a-vis the presence of a third firm.

With three firms and no NPE, licensing proceeds as in section (4.1), modified to take account of the additional firm. Denoting the firms i, j, k, let the (reduced form) profit for i, as a function of cost be $\pi^1(x_i, x_j, x_k)$. In this case, the maximal revenue from profit plus license fees to i when i becomes the patent holder becomes :

$$\begin{aligned}
\psi_3(x_i) &= \max_{f\leq 1-x_i} \{\pi^i(x_i, x_i+f, x_i+f) + [\pi^j(x_i, x_i+f, x_i+f) - \pi^j(x_i,1,1)] + f\hat{q}_j(f,x_i)\} \\
&\qquad\qquad + [\pi^k(x_i, x_i+f, x_i+f) - \pi^k(x_i,1,1)] + f\hat{q}_k(f,x_i)\} \\
&= \max_{f\leq 1-x_i} \left\{[p^i\hat{q}_i - x_i\hat{q}_i] + [p^j\hat{q}_j - x_i\hat{q}_j] + [p^k\hat{q}_k - x_i\hat{q}_k]\right\} - \pi^j(x_i,1,1) - \pi^k(x_i,1,1) \\
&= \psi_3^*(x_i) - \pi^j(x_i,1,1) - \pi^k(x_i,1,1), \ \psi_3^*(x_i) \stackrel{\text{def}}{=} \max_{f\leq 1-x_i} \left\{[p^i\hat{q}_i - x_i\hat{q}_i] + [p^j\hat{q}_j - x_i\hat{q}_j] + [p^k\hat{q}_k - x_i\hat{q}_k]\right\}
\end{aligned} \tag{18}$$

Here, $\hat{q}_r$ is the equilibrium output level of firm r when the cost structure is $(c_i, c_j, c_k) = (x_i, x_i+f, x_i+f)$.

As in section (4.3), when both j and k invest, let $A_3(x_k)$ be the expected value to firm i from investing and $C_3(x_k)$ be the payoff if i does not invest.

$$A_3(x_k) = \int_0^{x_k}\int_{x_i}^1 \psi_3(x_i)dF_j dF_i + \int_0^{x_k}\int_{x_j}^1 \pi^i(1,x_j,1)dF_i dF_j + \int_{x_k}^1\int_{x_k}^1 \pi^i(1,1,x_k)dF_j dF_i \tag{19}$$

$$C_3(x_k) = \int_0^{x_k}\int_0^1 \pi^i(1,x_j,1)dF_i dF_j + \int_{x_k}^1\int_0^1 \pi^i(1,1,x_k)dF_j dF_i \tag{20}$$

Let $A_3 = \int_0^1 A_3(x_k)dF_k$ and $C_3 = \int_0^1 C_3(x_k)dF_k$.

Next, suppose that k does not invest (but j does). Then the payoffs to i from investing (B_3) and not investing (D_3) respectively are:

$$\begin{aligned}
B_3 &= \int_0^1\int_{x_i}^1 \psi_3(x_i)dF_j dF_i + \int_0^1\int_0^{x_i} \pi^i(1,x_j,1)dF_j dF_i \\
D_3 &= \int_0^1 \pi^i(1,x_j,1)dF_j
\end{aligned}$$

Finally, if neither j or k invest, then the payoffs to i from investing (E) and not investing (F) respectively are:

$$E_3 = \int_0^1 \psi_3^i(x_i,1,1)dF_i, \qquad F_3 = \pi^i(1,1,1)$$

In this environment, all three firms will in equilibrium invest in R&D if $\rho < A_3 - C_3$; two firms will invest if $A_3 - C_3 < \rho < B_3 - D_3$, one firm will invest if $B_3 - D_3 < \rho < E_3 - F_3$, and no firm will invest of $E_3 - F_3 < \rho$. This is the case, regardless of whether the firms supply differentiated markets or not. However, as mentioned, with differentiated commodities an additional market (with demand p^k) is added in this model making interpretation difficult. A special case occurs when all firms share a market, so that

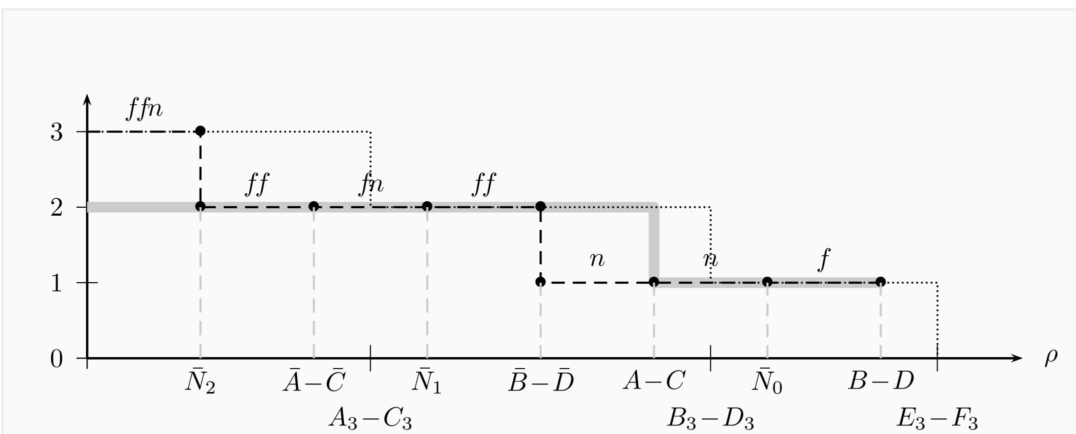
$p^i = p^j = p^k = P$. In what follows, take the linear demand model as discussed in section (5).

Proposition 8: Suppose that three firms supply the market with demand $P = a - bQ$. Then

$A_3 - C_3$	$B_3 - D_3$	$F_3 - E_3$
$\frac{1}{480b}(-112 + 115a)$	$\frac{1}{96b}(-29 + 30a)$	$\frac{1}{48b}(-20 + 21a)$

Proposition 9: Let $I_3(\rho)$ be the number of firms investing in equilibrium at cost ρ. Then, for all ρ, $I_3(\rho) \geq I_0(\rho) \geq I_N(\rho)$ (where $I_0(\rho)$ is investment in the two firm case and $I_N(\rho)$ is investment with the two firms and NPE).

Proof: The proof follows directly from comparison of the various thresholds, depicted in figure (9). Here, the dotted line, ··············, represents the number of investing firms when there are three firms operating in the market.



■

Figure 9: Investment, including 3 firm case.

The impact of an NPE on the overall level of investment and innovation depends on parameter values (profit functions and technology distributions.) The presence of an NPE always raises investment when investment cost is low. Also, the maximum cost at which investment would take place is never lower with an NPE present.

6 Conclusion

The impact of an NPE on the overall level of investment and innovation depends on parameter values (profit functions and technology distributions.) The presence of an NPE always raises investment when investment cost is low. Also, the maximum cost at which investment would take place is never lower with an NPE present. This is because the NPE either invests at cost levels that firms would not invest at if the NPE were absent; or at high cost levels the NPE simply drops out and the firms behave as they would if there where no NPE. At intermediate costs, the impact on investment and innovation is ambiguous and requires a case by case evaluation characterized by threshold levels of entry and exit from investment.

Appendices

A Monopoly: Appendix for section 3

Proposition 1: With an NPE, both firm and NPE invest when $\rho < \underline{\rho}$. On the region $[\underline{\rho}, \bar{\rho}]$ only one invests with the firm investing if $E_F > E_N$ and the NPE investing when $E_F < E_N$. On the region $[\bar{\rho}, \rho^*]$ only one invests. If $\bar{\rho} < \rho_*$, then on the region $[\bar{\rho}, \rho_*]$ only one invests and it may be either the firm or the NPE. On the region $[\rho_*, \rho^*]$ the entity with the larger value of $\{S_F, S_N\}$ is the sole investor.

Proof: Consider first the case where both invest. Conditional on z, the expected payoffs to the firm and NPE are:

$$\begin{aligned} V_F(z) &= \int_0^1 V_F(x,z)dF(x) \; = \int_0^z \pi(x)dF(x) + \int_z^1 \pi(1)dF(x) \\ V_N(z) &= \int_0^1 V_N(x,z)dF(x) \; = \int_z^1 [\pi(z)-\pi(1)]dF(x) = [\pi(z)-\pi(1)][1-F(z)] \end{aligned}$$

Given z, if $x < z$ then the firm gets profit $\pi(x)$ while the NPE gets nothing: conversely, when $x > z$, the NPE extracts the full surplus $\pi(x) - \pi(1)$ and the firm gets $\pi(1)$. Thus, the unconditional expected payoffs are $\bar{V}_F = E\{V_F(z)\}$ and $\bar{V}_N = E\{V_N(z)\}$. So,

$$\bar{V}_F = \int_0^1 \int_0^z \pi(x)dF(x)dG(z) + \int_0^1 \int_z^1 \pi(1)dF(x)dG(z) \tag{21}$$

and

$$\begin{aligned} \bar{V}_N &= \int_0^1 \int_z^1 \pi(z)dF(x)dG(z) - \int_0^1 \int_z^1 \pi(1)dF(x)dG(z) && (22) \\ &= \int_0^1 [\pi(z)-\pi(1)][1-F(z)]dG(z) && (23) \end{aligned}$$

The net gain to the firm from investing (given the NPE also invests) is $\bar{V}_F - \pi(1)$:

$$\bar{V}_F - \pi(1) = \int_0^1 \int_0^z [\pi(x)-\pi(1)]dF(x)dG(z) \tag{24}$$

From remark (A.2) below, for any function $h(x)$, $\int_0^1 \int_0^z h(x)dF(x)dG(z) = \int_0^1 h(x)[1-G(x)]dF(x)$. Applying this to equation (24),

$$\bar{V}_F - \pi(1) \quad = \quad \int_0^1 [\pi(x) - \pi(1)][1 - G(x)]dF(x) \tag{25}$$

These calculations give two exit thresholds. Given the firm chooses to invest, the NPE would exit when ρ is larger than $E_N \stackrel{\text{def}}{=} \bar{V}_N$, and given that the NPE chooses to invest, the firm would exit at whenever ρ is larger than $E_F \stackrel{\text{def}}{=} \bar{V}_F - \pi(1)$. Exit thresholds are given by

$$\underline{\rho} = \min\{E_F, E_G\} \text{ and } \bar{\rho} = \max\{E_F, E_G\}$$

For $\rho < \underline{\rho}$, both firm and NPE invest; between $\underline{\rho}$ and $\bar{\rho}$ the entity with the larger exit threshold invests alone.

These exit thresholds may be ordered, if F and G are ordered (in stochastic dominance terms). Letting $\psi(z) = \pi(z) - \pi(1)$, $\psi(z)$ is decreasing in z, as are $[\pi(z) - \pi(1)][1 - F(z)]$ and $[\pi(z) - \pi(1)][1 - G(z)]$. Suppose that $F \succeq G$ so that $F(z) \leq G(z)$ and $[1 - F(z)] \geq [1 - G(z)]$. Then

$$E_N = \int_0^1 \psi(z)[1 - F(z)]dG(z) \geq \int_0^1 \psi(z)[1 - G(z)]dG(z) \geq \int_0^1 \psi(z)[1 - G(z)]dF(z) = E_F$$

where the last inequality follows because $\psi(z)[1 - G(z)]$ is decreasing and $F \succeq G$. Therefore, if $F \succeq G$, then $E_N \geq E_F$. Conversely, if $G \succeq F$, then $E_F \geq E_N$.

If the firm alone invests, it obtains expected profit of $\bar{\pi}_F = \int_0^1 \pi(x)dF(x)$ for a profit net of investment cost of $\bar{\pi}_F - \rho$, whereas not investing yields $\pi(1)$. The gain from investing, when investing alone is $S_F = \bar{\pi}_F - \pi(1)$. If the NPE alone invests, the NPE extracts the full surplus, $\pi(z) - \pi(1)$, at technology z and the expected benefit to the NPE is $S_N = \int_0^1 \pi(z)dG - \pi(1) = \bar{\pi}_G - \pi(1)$. If $F \succeq G$, $S_N \geq S_F$ and if $G \succeq F$, $S_F \geq S_N$.

Comparing V_F and $\bar{\pi}_F$:

$$\begin{aligned}
\bar{V}_F \quad &= \quad \int_0^1 \int_0^z \pi(x)dF(x)dG(z) + \int_0^1 \int_z^1 \pi(1)dF(x)dG(z) \\
\bar{\pi}_F \quad &= \quad \int_0^1 \int_0^z \pi(x)dF(x)dG(z) + \int_0^1 \int_z^1 \pi(x)dF(x)dG(z)
\end{aligned}$$

so

$$\begin{aligned}\bar{\pi}_F - \bar{V}_F &= \int_0^1 \int_z^1 [\pi(x) - \pi(1)] dF(x) dG(z) \\ &= \int_0^1 [\pi(x) - \pi(1)] G(x) dF(x) \\ &= \int_0^1 \psi(x) G(x) dF(x)\end{aligned}$$

The gap between the firms' exit thresholds S_F and E_F is:

$$S_F - E_F = [\bar{\pi}_F - \pi(1)] - [\bar{V}_F - \pi(1)] = \bar{\pi}_F - \bar{V}_F = \int_0^1 [\pi(x) - \pi(1)] G(x) dF(x) = \int_0^1 \psi(x) G(x) dF(x)$$

Similarly, with

$$\begin{aligned}\bar{V}_N &= \int_0^1 \int_0^x [\pi(z) - \pi(1)] dG(z) dF(x) + \int_0^1 \int_x^1 0 \cdot dG(z) dF(x) \\ \bar{\pi}_G - \pi(1) &= \int_0^1 \int_0^x [\pi(z) - \pi(1)] dG(z) dF(x) + \int_0^1 \int_x^1 [\pi(z) - \pi(1)] dG(z) dF(x)\end{aligned}$$

so that $[\bar{\pi}_G - \pi(1)] - \bar{V}_N = \int_0^1 \int_x^1 [\pi(z) - \pi(1)] dG(z) dF(x) = \int_0^1 [\pi(z) - \pi(1)] F(z) dG(z)$. Therefore, the gap between the NPE's exit thresholds S_N and E_N is

$$S_N - E_N = \int_0^1 [\pi(z) - \pi(1)] F(z) dG(z) = \int_0^1 \psi(z) F(z) dG(z) \qquad \blacksquare$$

In general, there is no connection between the ordering of the pairs (E_N, E_F) and (S_N, S_F), as the following example illustrates.

EXAMPLE A.1: Let $\pi(x) - \pi(1) = 1 - x$, $F(x) = x$ and

$$G(x) = \begin{cases} 2x(1-x), & x \leq \frac{1}{2} \\ 1 - 2x(1-x), & x > \frac{1}{2} \end{cases}$$

F second order stochastically dominates G. The following calculations show that $S_F = S_N$ and $E_F < E_N$. To see that $S_K = S_N$:

$$S_F = \int_0^1 \psi(x) dF(x) \quad = \quad \int_0^1 (1-x) dx = \frac{1}{2}$$

$$
\begin{aligned}
S_N = \int_0^1 \psi(x)dG(x) &= \int_0^{\frac{1}{2}} (1-x)(2-4x)dx + \int_{\frac{1}{2}}^1 (1-x)(-2+4x)dx \\
&= \int_0^{\frac{1}{2}} (1-x)(2-4x)dx - \int_{\frac{1}{2}}^1 (1-x)(2-4x)dx \\
&= \frac{5}{12} - (-\frac{1}{12}) = \frac{1}{2}
\end{aligned}
$$

Using the fact that $E_N = S_N - \int_0^1 \psi(z)F(z)dG(z)$,

$$
\begin{aligned}
E_N &= S_N - \big(\int_0^{\frac{1}{2}} (1-x)x[2-4x]dx + \int_{\frac{1}{2}}^1 (1-x)x[-2+4x]dx \big) \\
&= S_N - \big(\int_0^{\frac{1}{2}} (1-x)x[2-4x]dx - \int_{\frac{1}{2}}^1 (1-x)x[2-4x]dx \big) \\
&= S_N - \big(\frac{1}{16} - [-\frac{1}{16}] \big) = S_N - \frac{1}{8} = \frac{3}{8}
\end{aligned}
$$

Similarly, $E_F = S_F - \int_0^1 \psi(x)G(x)dF(x)$,

$$
\begin{aligned}
E_F &= S_F - \big(\int_0^{\frac{1}{2}} (1-x)2x(1-x)dx + \int_{\frac{1}{2}}^1 (1-x)[1-2x(1-x)]dx \big) \\
&= S_F - \big(\frac{11}{96} + \frac{7}{96} \big) = S_F - \frac{18}{96} = \frac{1}{2} - \frac{3}{16} = \frac{5}{16}
\end{aligned}
$$

So, $E_N = \frac{6}{16}$ and $E_F = \frac{5}{16}$. For the net profit function ψ, $E_F < E_N$ and $S_F = S_N$.

If instead, profit were given by $\hat{\pi}(x) - \hat{\pi}(1) = \hat{\psi}(x) = 1 - x^2$, then

$$
\begin{aligned}
\hat{S}_F &= \int_0^1 (1-x^2)dx = \frac{2}{3} = \frac{32}{48} \\
\hat{S}_E &= \int_0^{\frac{1}{2}} (1-x^2)(2-4x)dx + \int_{\frac{1}{2}}^0 (1-x^2)[-2+4x]dx \\
&= \int_0^{\frac{1}{2}} (1-x^2)(2-4x)dx - \int_{\frac{1}{2}}^0 (1-x^2)[2-4x]dx \\
&= \frac{23}{48} - (-\frac{7}{48}) = \frac{30}{48}
\end{aligned}
$$

◇

Therefore, for the net profit function $\psi_\alpha = \psi + \alpha\hat{\psi}$, with α small, the corresponding exit thresholds (with α superscripts) satisfy: $E_F^\alpha < E_N^\alpha$ and $S_F^\alpha > S_N^\alpha$.

In contrast, if profit is given by $\tilde{\pi}(x) - \tilde{\pi}(1) = \tilde{\psi}(x) = 1 - \sqrt{x}$, then $\tilde{S}_F = \frac{1}{3}$ while $\tilde{S}_N = \frac{11}{15} - \frac{4}{15}\sqrt{2} \approx 0.356$.

REMARK A.1: Note $\int_0^z [1-F(x)]dF(x) = \int_0^z dF(x) - \int_0^z F(x)dF(x) = F(z) - \frac{1}{2}F(z)^2$, so that $\int_0^1 [1-F(x)]dF(x) = \frac{1}{2}$. Also, letting $G = F + (G-F)$,

$$\gamma = \int_0^1 [1-F(z)]dG(z) = \int_0^1 [1-F(z)]dF(z) + \int_0^1 [1-F(z)](dG-dF) = \frac{1}{2} - \int_0^1 [F(z)-1](dG-dF)$$

So, if G first order stochastically dominates F, then $\gamma < \frac{1}{2}$ and if F first order stochastically dominates G, then $\gamma > \frac{1}{2}$. Note that when G stochastically dominates F, draws from G tend to be higher — meaning lower quality technology. When $F = G$, $\gamma = \frac{1}{2}$.

REMARK A.2: Let $h(x)$ be a function on $[0,1]$ and F, G continuous distributions on $[0,1]$. Then

$$\int_0^1 \int_0^z h(x)dF(x)dG(z) = \int_0^1 h(x)[1-G(x)]dF(x)$$

To see this, consider discrete approximations to the integral on a grid $\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}\} = \{\delta_0, \delta_1, \dots, \delta_n\}$. Define $f_k = F(\frac{k}{n}) - F(\frac{k-1}{n})$ with $F(0) = 0$, $k = 1, \dots, n$. Similarly, $g_k = G(\frac{k}{n}) - G(\frac{k-1}{n})$. Then

$$\int_0^1 \int_0^z h(x)dF(x)dG(z) \;\approx\; \sum_{z=\delta_1}^{\delta_n} \sum_{x=\delta_1}^{z} h(x)f_x g_z \;\approx\; \int_0^1 h(x)[1-G(x)]dF(x)$$

To see this, consider:

$$\begin{aligned}
\sum_{z=\delta_1}^{\delta_n} \sum_{x=\delta_1}^{z} h(x)f_x g_z \quad = \quad & h(\delta_1)f_{\delta_1}g_{\delta_1} \\
& + h(\delta_1)f_{\delta_1}g_{\delta_2} + h(\delta_2)f_{\delta_2}g_{\delta_2} \\
& + h(\delta_1)f_{\delta_1}g_{\delta_3} + h(\delta_2)f_{\delta_2}g_{\delta_3} + h(\delta_3)f_{\delta_3}g_{\delta_3} \\
& + h(\delta_1)f_{\delta_1}g_{\delta_4} + h(\delta_2)f_{\delta_2}g_{\delta_4} + h(\delta_3)f_{\delta_3}g_{\delta_4} + h(\delta_4)g_{\delta_4}f_{\delta_4} \\
& + \cdots \\
= \quad & h(\delta_1)f_{\delta_1} + h(\delta_2)f_{\delta_2}[G(\delta_n) - G(\delta_1)] + h(\delta_3)f_{\delta_3}[G(\delta_n) - G(\delta_2)] + \cdots \\
= \quad & h(\delta_1)f_{\delta_1} + h(\delta_2)f_{\delta_2}[1 - G(\delta_1)] + h(\delta_3)f_{\delta_3}[1 - G(\delta_2)] + \cdots
\end{aligned}$$

so,

$$\int_0^1 \int_0^z h(x)dF(x)dG(z) \quad = \quad \int_0^1 h(z)[1-G(z)]dF(z)$$

Similarly, $\int_0^1 \int_z^1 h(x)dF(x)dG(z) = \int_0^1 h(x)G(x)dF(x)$. To see this, consider:

$$\begin{aligned}
\sum_{z=\delta_1}^{\delta_n} \sum_{x=z}^{\delta_n} h(x) f_x g_z \quad = \quad & h(\delta_1) f_{\delta_1} g_{\delta_1} + h(\delta_2) f_{\delta_2} g_{\delta_1} + h(\delta_3) f_{\delta_3} g_{\delta_1} + \cdots + h(\delta_n) f_{\delta_n} g_{\delta_1} \\
& + h(\delta_2) f_{\delta_2} g_{\delta_2} + h(\delta_3) f_{\delta_3} g_{\delta_2} + \cdots + h(\delta_n) f_{\delta_n} g_{\delta_2} \\
& + h(\delta_3) f_{\delta_3} g_{\delta_3} + \cdots + h(\delta_n) f_{\delta_n} g_{\delta_3} \\
& \ddots \quad \vdots \\
& h(\delta_n) f_{\delta_n} g_{\delta_n} \\
= \quad & h(\delta_1) f_{\delta_1} G(\delta_1) + h(\delta_2) f_{\delta_2} G(\delta_2) + h(\delta_3) f_{\delta_3} G(\delta_3) + \cdots
\end{aligned}$$

Which is approximately $\int_0^1 h(x)G(x)dF(x)$.

B Licensing: Appendix for section 4.1.

The following discussion confirms the contents of remark (4.1). Lemma (1) show that $R(f, x_i)$ is decreasing in f under general conditions. Lemma (2) shows that the optimal license fee (royalty) set by a firm is strictly positive, and lemma (3) shows this is also true when the NPE is licensor.

LEMMA 1: Suppose that the goods are substitutes meaning that $p_i^j < 0$. If increasing the royalty fee reduces the equilibrium output of j, $\frac{\partial \hat{q}_j}{\partial f} < 0$, and if the goods are strategic substitutes, so that the best response of i is decreasing in j's output, then a licence fee of 0 is a global maximum for R, with license revenue $R(0)$.

$$R(0; x_i) \geq \max_{f \leq 1 - x_i} R(f; x_i)$$

$\diamond$

PROOF: Let $(\hat{q}_i, \hat{q}_j)$ be the equilibrium quantities (which depend on (f, x_i), $\hat{q}_i(f, x_i)$, and $\hat{q}_i(f, x_i)$). The corresponding equilibrium profits are: $\pi^i(x_i, x_i + f) = \pi^i(x_i, x_i + f; \hat{q}_i, \hat{q}_j)$, $\pi^j(x_i, x_i + f) = \pi^j(x_i, x_i + f; \hat{q}_i, \hat{q}_j)$.

Given licensing by i at fee f, the choices $\hat{q}_i$ and $\hat{q}_j$ satisfy the first order conditions:

$$\hat{q}_i p_i^i + p^i - x_i = 0, \qquad \hat{q}_j p_j^j + p^j - (x_i + f) = 0. \tag{26}$$

(With notation $p^k_r = p^k_r(q_i, q_j) = \frac{\partial p^k(q_i,q_j)}{dq_r}$, $k, r = i, j$).

The direct impact of an increase in f is to alter the marginal cost of the licensee. This affects the profit of both licensor and licensee:

$$\begin{aligned}
\pi^i_f = \frac{\partial \pi^i(x_i, x_i + f)}{\partial f} &= \hat{q}_i p^i_i \frac{\partial \hat{q}_i}{\partial f} + p^i \frac{\partial \hat{q}_i}{\partial f} + \hat{q}_i p^i_j \frac{\partial \hat{q}_j}{\partial f} - x_i \frac{\partial \hat{q}_i}{\partial f} \\
&= [\hat{q}_i p^i_i + p^i - x_i] \frac{\partial \hat{q}_i}{\partial f} + \hat{q}_i p^i_j \frac{\partial \hat{q}_j}{\partial f} \\
&= \hat{q}_i p^i_j \frac{\partial \hat{q}_j}{\partial f} \qquad (27) \\
\pi^j_f = \frac{\partial \pi^j(x_i, x_i + f)}{\partial f} &= \hat{q}_i p^j_j \frac{\partial \hat{q}_j}{\partial f} + p^j \frac{\partial \hat{q}_j}{\partial f} + \hat{q}_j p^j_i \frac{\partial \hat{q}_i}{\partial f} - (x_i + f) \frac{\partial \hat{q}_j}{\partial f} - \hat{q}_j \\
&= [\hat{q}_i p^j_j + p^j - (x_i + f)] \frac{\partial \hat{q}_j}{\partial f} + \hat{q}_j p^j_i \frac{\partial \hat{q}_i}{\partial f} - \hat{q}_j \\
&= \hat{q}_j p^j_i \frac{\partial \hat{q}_i}{\partial f} - \hat{q}_j \qquad (28)
\end{aligned}$$

The impact of varying the royalty fee is (using equation (28)):

$$\begin{aligned}
R_f(f; x_i) &= \pi^j_f(x_i, x_i + f) + \hat{q}_j(f, x_i) + f \frac{\partial \hat{q}_j}{\partial f} \\
&= \hat{q}_j(f, x_i) p^j_i \frac{\partial \hat{q}_i}{\partial f} - \hat{q}_j(f, x_i) + \hat{q}_j(f, x_i) + f \frac{\partial \hat{q}_j}{\partial f} \\
&= \hat{q}_j(f, x_i) p^j_i \frac{\partial \hat{q}_i}{\partial f} + f \frac{\partial \hat{q}_j}{\partial f} \qquad (29)
\end{aligned}$$

Since the goods are strategic substitutes (best response function decreasing), an increase in f lowers $\hat{q}_j(f; x_i)$ and hence raises $\hat{q}_i$: $\frac{\partial \hat{q}_i}{\partial f} > 0$. Since $\frac{\partial \hat{q}_j}{\partial f} < 0$ and $p^i_j < 0$, $R_f(f; x_i) < 0$. For $f > 0$,

$$R(f; x_i) - R(0 : x_i) = \int_0^f R_f(\tilde{f}; x_i) d\tilde{f} < 0 \qquad \blacksquare$$

These calculations determine the optimal license fee to maximize direct revenue. Considering the overall impact of a licensing fee on the firms operating profit plus licensing revenue, the optimal choice of f is generally positive: $\hat{\pi}(x_i, f)$ is maximized at some $f > 0$because the positive license fee contracts the competitors output. This is the content of lemma (2).

LEMMA 2: The optimal license fee to maximize the sum of profit and licensing revenue is strictly positive provided the impact on aggregate output of a fee increase at 0 is negative. $\diamond$

PROOF: Differentiating $\hat{\pi}^i$ with respect to f (using equations (27) and (29)),

$$\begin{aligned}\hat{\pi}^i_f = \frac{\partial \hat{\pi}^i}{\partial f} &= \pi^i_f + R_f \\ &= p^i_j \hat{q}_i \frac{\partial \hat{q}_j}{\partial f} + p^j_i \hat{q}_j \frac{\partial \hat{q}_i}{\partial f} + f \frac{\partial \hat{q}_j}{\partial f}\end{aligned}$$

Symmetry, $p^i(a,b) = p^j(b,a)$, implies that $p^i_j(a,b) = p^j_i(b,a)$. At $f = 0$, with licensing both firms have marginal cost x_i, and choose the same output, so let $\hat{q} = \hat{q}_i = \hat{q}_j$. With $\delta \stackrel{\text{def}}{=} p^i_j(\hat{q},\hat{q}) = p^j_i(\hat{q},\hat{q}) < 0$,

$$\hat{\pi}^i_f \Big|_{f=0} = \hat{q} \cdot \delta(\frac{\partial \hat{q}_j}{\partial f} + \frac{\partial \hat{q}_i}{\partial f})$$

Provided aggregate output is decreasing in f, $(\frac{\partial \hat{q}_j}{\partial f} + \frac{\partial \hat{q}_i}{\partial f} < 0)$, $\hat{\pi}^i_f > 0$ at $f = 0$. ■

Finally, the optimal licensing behavior for the NPE requires a positive royalty fee (because it curtails output and raises profit which can be transferred to the NPE).

LEMMA 3: For the Non-practising entity, the optimal royalty fee is positive. ◇

PROOF: The impact of an increase in f on i's profit is:

$$\begin{aligned}\frac{\partial \pi^i(f)}{\partial f} &= p^i_i \frac{\partial \tilde{q}_i}{\partial f} + \tilde{q}_i p^i_i \frac{\partial \tilde{q}_i}{\partial f} + \tilde{q}_i p^i_j \frac{\partial \tilde{q}_j}{\partial f} - (z+f)\frac{\partial \tilde{q}_i}{\partial f} - \tilde{q}_i \\ &= [p^i_i + \tilde{q}_i p^i_i - (z+f)]\frac{\partial \tilde{q}_i}{\partial f} + \tilde{q}_i p^i_j \frac{\partial \tilde{q}_j}{\partial f} - \tilde{q}_i \\ &= \tilde{q}_i p^i_j \frac{\partial \tilde{q}_j}{\partial f} - \tilde{q}_i \end{aligned} \tag{30}$$

Maximizing $N(z; f)$ (using equation 30):

$$\begin{aligned}N_f(z;f) &= \tilde{q}_i p^i_j \frac{\partial \tilde{q}_j}{\partial f} - \tilde{q}_i + \tilde{q}_i + f\frac{\partial \tilde{q}_i}{\partial f} + \tilde{q}_j p^j_i \frac{\partial \tilde{q}_i}{\partial f} - \tilde{q}_j + \tilde{q}_j + f\frac{\partial \tilde{q}_j}{\partial f} \\ &= \tilde{q}_i p^i_j \frac{\partial \tilde{q}_j}{\partial f} + f\frac{\partial \tilde{q}_i}{\partial f} + \tilde{q}_j p^j_i \frac{\partial \tilde{q}_i}{\partial f} + f\frac{\partial \tilde{q}_j}{\partial f}\end{aligned}$$

Thus,

$$N_f(z;0) = \tilde{q}_i p^i_j \frac{\partial \tilde{q}_j}{\partial f} + \tilde{q}_j p^j_i \frac{\partial \tilde{q}_i}{\partial f}$$

Since $p_i^j < 0, p_j^i < 0$ and $\frac{\partial \tilde{q}_i}{\partial f}, \frac{\partial \tilde{q}_j}{\partial f} < 0$, $N_f(0,z) > 0$. ■

C Equilibrium: Appendix for sections 4.2–4.4.

Proposition 2: Payoffs satisfy: $0 \leq A - C \leq B - D$. Let the investment cost be ρ. If $\rho < A - C$ there is a unique equilibrium where both firms invest. If $A - C < \rho < B - D$, then there are two equilibria where one or other firm invests. And if $B - D < \rho$ then the unique equilibria has neither firm invest.

Proof: This follows from proposition 3 (where the inequalities $0 \leq A - C \leq B - D$ are established). ■

Proposition 3: The differences $A(z) - C(z)$ and $B(z) - D(z)$ satisfy the following properties:

1. $A(0) - C(0) = B(0) - D(0) = 0$, $A - C = A(1) - C(1) \leq B(1) - D(1) = B - D$,
2. $0 \leq A'(z) - C'(z) \leq B'(z) - D'(z)$.

It follows from 1 and 2 that $A(z) - C(z) \leq B(z) - D(z)$ for all z.

Proof: Consider $A(z) - C(z)$. From equations (14) and (16):

$$\begin{aligned} A(z) - C(z) &= \int_0^z \int_{x_i}^1 \psi(x_i) dF_j dF_i + \int_0^z \int_{x_i}^1 \pi^j(x_i, 1) dF_j dF_i + \int_z^1 \int_z^1 \pi^i(1,1) dF_j dF_i \\ &\qquad - \int_0^z \int_0^1 \pi^j(x_i, 1) dF_j dF_i - \int_z^1 \int_0^1 \pi^i(1,1) dF_j dF_i \\ &= \int_0^z \int_{x_i}^1 \psi(x_i) dF_j dF_i - \int_0^z \int_0^{x_i} \pi^j(x_i, 1) dF_j dF_i - \int_z^1 \int_0^z \pi^i(1,1) dF_j dF_i \qquad (31) \end{aligned}$$

And, from equation (15), $B(z) - D(z)$:

$$\begin{aligned} B(z) - D(z) &= \int_0^z \int_0^1 \psi(x_i) dF_j dF_i + \int_z^1 \int_0^1 \pi^i(1,1) dF_j dF_i - \int_0^1 \int_0^1 \pi^i(1,1) dF_j dF_i \\ &= \int_0^z \int_0^1 \psi(x_i) dF_j dF_i - \int_0^z \int_0^1 \pi^i(1,1) dF_j dF_i \qquad (32) \end{aligned}$$

Setting $z = 0$, $A(0) - C(0) = B(0) - D(0) = 0$. Setting $z = 1$,

$$\begin{aligned} A(1) - C(1) &= \int_0^1 \int_{x_i}^1 \psi(x_i) dF_j dF_i - \int_0^1 \int_0^{x_i} \pi^j(x_i, 1) dF_j dF_i \qquad (33) \\ &= \int_0^1 \int_{x_i}^1 \psi(x_i) dF_j dF_i - \int_0^1 \int_{x_i}^1 \pi^i(1, x_j) dF_j dF_i \qquad (34) \\ &= A - C \qquad (35) \end{aligned}$$

$$
\begin{aligned}
B(1) - D(1) &= \int_0^1 \int_0^1 \psi(x_i) dF_j dF_i - \int_0^1 \int_0^1 \pi^i(1,1) dF_j dF_i && (36)\\
&= B - D && (37)
\end{aligned}
$$

(where (34) follows from $\int_0^1 \int_0^{x_i} \pi^j(x_i, 1) dF_j dF_i = \int_0^1 \int_{x_i}^1 \pi^i(1, x_j) dF_j dF_i$.)

Differentiating with respect to z,

$$
\begin{aligned}
A'(z) - C'(z) &= \psi(z)[1 - F(z)]f(z) - \pi^j(z,1)F(z)f(z) - \pi^i(1,1)\{[1 - F(z)]f(z) - F(z)f(z)\}\\
&= [\psi(z) - \pi^i(1,1)][1 - F(z)]f(z) + [\pi^i(1,1) - \pi^j(z,1)]F(z)f(z)\\
&= [\psi(z) - \pi^i(1,1)][1 - F(z)]f(z) + [\pi^j(1,1) - \pi^j(z,1)]F(z)f(z)\\
&\geq 0
\end{aligned}
$$

using the fact that $[\psi(z) - \pi^i(1,1)] \geq 0$ and $[\pi^j(1,1) - \pi^j(z,1)] \geq 0$. Next,

$$
\begin{aligned}
B(z) - D(z) &= \int_0^z \psi(x_i) dF_j dF_i - \int_0^z \int_0^1 \pi^i(1,1) dF_j dF_i\\
&= \int_0^z \psi(x_i) dF(x_i) - \pi^i(1,1)F(x_i)
\end{aligned}
$$

Thus,

$$
B'(z) - D'(z) = \psi(z)f(z) - \pi^i(1,1)f(z) = [\psi(z) - \pi^i(1,1)]f(z) \geq 0
$$

Finally, comparing $A'(z) - C'(z)$ with $B'(z) - D'(z)$,

$$
\begin{aligned}
A'(z) - C'(z) &= [\psi(z) - \pi^i(1,1)][1 - F(z)]f(z) + [\pi^j(1,1) - \pi^j(z,1)]F(z)f(z)\\
&= [\psi(z) - \pi^i(1,1)]f(z) - [\psi(z) - \pi^i(1,1)]F(z)f(z) + [\pi^j(1,1) - \pi^j(z,1)]F(z)f(z)\\
&= B'(z) - D'(z) - [\psi^*(z) - \pi^j(z,1) - \pi^i(1,1)]F(z)f(z) + [\pi^j(1,1) - \pi^j(z,1)]F(z)f(z)\\
&= B'(z) - D'(z) - [\psi^*(z) - \pi^i(1,1)]F(z)f(z) + \pi^j(1,1)F(z)f(z)\\
&= B'(z) - D'(z) + [2\pi^i(1,1) - \psi^*(z)]F(z)f(z)\\
&\leq B'(z) - D'(z)
\end{aligned}
$$

■

PROPOSITION 4: Equilibrium investment as ρ increases is as follows:

1. For $\rho \in [0, \bar{A} - \bar{C}]$:

   (a) If $\rho \leq \min\{\bar{N}_2, \bar{A} - \bar{C}\}$ the investment structure is ffn.

   (b) If $\min\{\bar{N}_2, \bar{A} - \bar{C}\} < \rho \leq \bar{A} - \bar{C}$ the investment structure is ff.

2. For $\rho \in (\bar{A} - \bar{C}, \bar{B} - \bar{D}]$:

   (a) If $\bar{A} - \bar{C} < \rho \leq \min\{\bar{N}_1, \bar{B} - \bar{D}\}$ the investment structure is fn.

   (b) If $\min\{\bar{N}_1, \bar{B} - \bar{D}\} < \rho \leq \min\{\max\{\bar{N}_1, A - C\}, \bar{B} - \bar{D}\}$ the investment structure is ff.

   (c) If $\max\{\bar{N}_1, A - C\} < \rho \leq \bar{B} - \bar{D}$, then investment structure is f.

3. For $\rho \in (\bar{B} - \bar{D}, B - D]$:

   (a) If $\bar{B} - \bar{D} < \rho \leq \max\{\bar{B} - \bar{D}, \bar{N}_0\}$ the investment structure is n.

   (b) If $\max\{\bar{B} - \bar{D}, \bar{N}_0\} < \rho \leq \max\{\max\{\bar{B} - \bar{D}, \bar{N}_0\}, A - C\}$ the investment structure is ff.

   (c) If $\max\{\max\{\bar{B} - \bar{D}, \bar{N}_0\}, A - C\} < \rho \leq \max\{\max\{\bar{B} - \bar{D}, \bar{N}_0\}, B - D\}$ the investment structure is f.

   (d) If $\max\{\max\{\bar{B} - \bar{D}, \bar{N}_0\}, B - D\} < \rho$ there is no investment.

4. For $\rho \in (B - D, \infty)$:

   - If $\rho > \max\{\bar{N}_0, B - D\}$, or there is no investment.
   - If $\bar{N}_0 > B - D$ the NPE alone invests for $\rho \in (B - D, \bar{N}_0)$.

PROOF: Consider these cases in turn, beginning with $\rho \in [0, \bar{A} - \bar{C}]$. If $\rho \leq \min\{\bar{N}_2, \bar{A} - \bar{C}\}$ then neither firm or the NPE will exit. For $\rho < \bar{A} - \bar{C}$ neither firm will exit, and the NPE will exit if and only if $\rho > \bar{N}_2$. If $\bar{N}_2 < \bar{A} - \bar{C}$, then for $\rho \in [0, \bar{N}_2)$ both firms and the NPE invest, while for $\rho \in [\bar{N}_2, \bar{A} - \bar{C})$ the two firms alone invest.

Next, consider the case where $\rho \in (\bar{A} - \bar{C}, \bar{B} - \bar{D}]$. If $\bar{N}_1 \leq \bar{A} - \bar{C}$ the condition $\bar{A} - \bar{C} < \rho \leq \min\{\bar{N}_1, \bar{B} - \bar{D}\}$ is vacuous. If $\bar{A} - \bar{C} < \bar{N}_1 \leq \bar{B} - \bar{D}$, then for $\rho \in (\bar{A} - \bar{C}, \bar{N}_1]$ the NPE invests, along with one firm. If $\bar{N}_1 \geq \bar{B} - \bar{D}$, then for $\rho \in (\bar{A} - \bar{C}, \bar{B} - \bar{D}]$ the NPE and one firm invests. If instead, $\bar{N}_1 < \bar{B} - \bar{D}$, the NPE will not invest if a least one firm chooses to invest and because $\rho \leq \bar{B} - \bar{C}$, at least one firm will invest. So, with $\bar{N}_1 < \rho \leq \bar{B} - \bar{C}$, the NPE will not invest and at least one firm will invest. Whether a second firm invests or not (in equilibrium) depends on the location of $A - C$. If $A - C \leq \bar{N}_1$, then just one firm invests for $\rho \in (\bar{N}_1, \bar{B} - \bar{C}]$ (and the NPE does not invest.) If $\bar{N}_1 < A - C \leq \bar{B} - \bar{D}$, then both firms will invest on $(\bar{N}_1, A-C]$ and just one firm will invest on $(A-C, \bar{B}-\bar{D}]$, provided $A-C < \bar{B}-\bar{D}$; if $A-C \geq \bar{B}-\bar{D}$ both firms invest on $(\bar{N}_1, \bar{B}-\bar{D}]$. (To clarify, when $\bar{N}_1 < \rho < A-C \leq \bar{B}-\bar{D}$, a decision to invest by the NPE will, on that subgame, lead to investment by a single firm and a negative payoff to the NPE, since $\rho > \bar{N}_1$. Hence, for such values of ρ, the NPE will no invest. However, since $\rho < A-C$, equilibrium of the subgame has both firms invest.) Figure (10) depicts the cases where both $\bar{N}_1$ and $A-C$

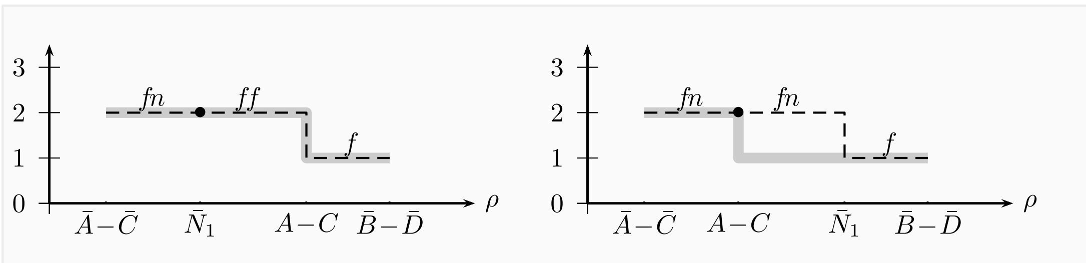


Figure 10: Investment on $(\bar{A}-\bar{C}, \bar{B}-\bar{D}]$.

lie in the interval $(\bar{A}-\bar{C}, \bar{B}-\bar{D}]$. In the figure, the dashed line indicates the number of investors and labels such as *fn* indicate that one firm (f) and the NPE (n) invest.

Considering the interval $(\bar{B}-\bar{D}, B-D]$, consider the two cases $A-C < \bar{B}-\bar{D}$ and $\bar{B}-\bar{D} < A-C$. For the first of these there are three possibilities for $\bar{N}_0$: $\bar{N}_0 < \bar{B}-\bar{D}$, $\bar{B}-\bar{D} < \bar{N}_0 < B-D$ and $\bar{N}_0 \geq B-D$. In the first case (where $A-C < \bar{B}-\bar{D}$), (i) if $N_0 \leq \bar{B}-\bar{D}$, then one firm alone invests for $\rho \in (\bar{B}-\bar{D}, B-D]$, (ii) if $\bar{B}-\bar{D} < \bar{N}_0 < B-D$, the NPE alone invests for $\rho \in (\bar{B}-\bar{D}, \bar{N}_0]$ and for $\rho \in (\bar{N}_0, B-D]$ one firm alone invests, (iii) if $\bar{N}_0 \geq B-D$ the NPE alone invests for $\rho \in (\bar{B}-\bar{D}, B-D]$. Thus, in this case, $A-C < \bar{B}-\bar{D}$, exactly one party invests for any $\rho \in (\bar{B}-\bar{D}, B-D]$.

In the second case, $\bar{B}-\bar{D} < A-C$ there are three possibilities, depending on the location of $\bar{N}_0$. (i) If $\bar{N}_0 < \bar{B}-\bar{D}$, then for $\rho \in (\bar{B}-\bar{D}, A-C]$, the two firms invest, the NPE does not, and for $\rho \in (A-C, B-D]$ a single firm invests. (ii) If $\bar{B}-\bar{D} < \bar{N}_0 < A-C$ then for $\rho \in (\bar{B}-\bar{D} < \bar{N}_0]$ the NPE alone invests; for $\rho \in (\bar{N}_0, A-C]$, both firms invest; and for $\rho \in (A-C, B-D]$ one firm alone invests. (iii) If $\bar{B}-\bar{D} < A-C < \bar{N}_0 \leq B-D$, the NPE alone invests when $\rho \in (\bar{B}-\bar{D}, \bar{N}_0]$ and with $\rho \in (\bar{N}_0, B-D]$ a single firm invests. (If $\bar{N}_0 > B-D$, the NPE alone invests for $\rho \in (\bar{B}-\bar{D}, \bar{N}_0]$.) Figure (11) depicts the cases where both $\bar{N}_0$ and $A-C$ lie in the interval $(\bar{B}-\bar{D}, B-D]$.

Finally, consider the interval $(B-D, \infty)$. If $\bar{N}_0 \leq B-D$, there is no investment. If $\bar{N}_0 > B-D$ the NPE alone invests for $\rho \in (B-D, \bar{N}_0]$ and there is no investment for $\rho \in (\bar{N}_0, \infty)$. ∎

Let $I_N(\rho)$ and $I_0(\rho)$ be total investment when the NPE is present ($I_N(\rho)$), and when the npe is not

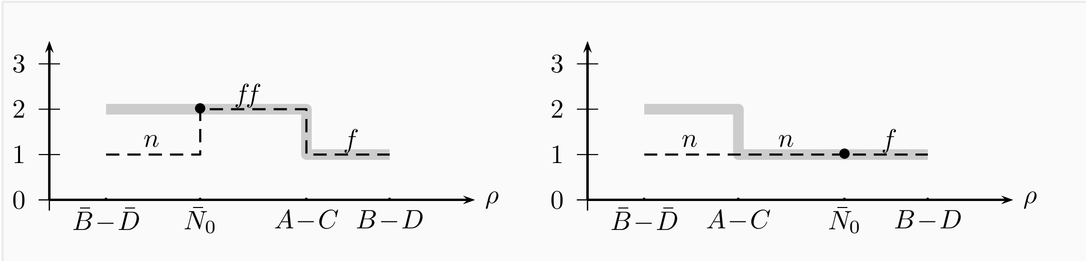


Figure 11: Investment on $(\bar{B} - \bar{D}, B - D]$.

present $(I_0(\rho))$. The following proposition characterizes the levels of investment in the two regimes.

Proposition 5: The impact of the NPE on investment (relative to the environment with just two firms) is as follows:

1. For $\rho \in [0, \bar{A} - \bar{C}]$, $I_N(\rho) \geq I_0(\rho)$, with strict inequality for $\rho \leq \bar{N}_2$.

2. For $\rho \in (\bar{A} - \bar{C}, \bar{B} - \bar{D}]$, $I_N(\rho) \geq I_0(\rho)$, with equality if $A - C \geq \min\{\bar{B} - \bar{D}, \bar{N}_1\}$. If $A - C < \min\{\bar{B} - \bar{D}, \bar{N}_1\}$ then $I_N(\rho) > I_0(\rho)$ for $\rho \in (A–C, \min\{\bar{B} - \bar{D}, \bar{N}_1\}]$. If $A–C < \min\{\bar{B} - \bar{D}, \bar{N}_1\} < \bar{B} - \bar{D}$, then $I_N(\rho) = I_0(\rho)$ for $(\bar{N}_1, \bar{B} - \bar{D}]$.

3. For $\rho \in (\bar{B} - \bar{D}, B - D]$, $I_N(\rho) \leq I_0(\rho)$, with equality if $\min\{A–C, \bar{N}_1\} \leq \bar{B} - \bar{D}$. If $\min\{A–C, \bar{N}_1\} > \bar{B} - \bar{D}$ then $I_N(\rho) < I_0(\rho)$ for $\rho \in (\bar{B} - \bar{D}, \min\{A–C, \bar{N}_1\}]$ and $I_N(\rho) = I_0(\rho)$ for $\rho \in (\min\{A–C, \bar{N}_1\}, B–D]$.

4. For $\rho \in (B - D, \infty)$, if $\bar{N}_0 \leq B - D$, then $I_N(\rho) = I_0(\rho) = 0$ and if $\bar{N}_0 > B - D$, $I_N(\rho) > I_0(\rho)$ for $\rho \in (B - D, \bar{N}_0]$ and $I_N(\rho) = I_0(\rho) = 0$ for $\rho \in (\bar{N}_0, \infty)$.

Proof: Consider the cases in turn:

1. For $\rho \in [0, \bar{A} - \bar{C}]$ if $\rho < \bar{N}_2$, then the NPE will invest, as do both firms. If $\bar{N}_2 < \bar{A} - \bar{C}$ the NPE will not invest and both firms will invest.

2. When $\rho \in (\bar{A} - \bar{C}, \bar{B} - \bar{D}]$ just one firm will invest if the NPE invests. Whether the NPE invests or not depends on the location of $\bar{N}_1$.

   - If $\bar{N}_1 > \bar{B} - \bar{D}$, then the NPE and exactly one firm will invest for ρ in the interval $(\bar{A} - \bar{C}, \bar{B} - \bar{D}]$. In contrast, in the environment with no NPE, as long as $A–C \geq \bar{B} - \bar{D}$, two firms will invest for all $\rho \in (\bar{A} - \bar{C}, \bar{B} - \bar{D}]$. So, in this case, the level of investment is the same with or without the presence of the NPE. If, however, $A–C < \bar{B} - \bar{D}$, then for ρ with $A–C < \rho < \bar{B} - \bar{D}$, exactly one firm will invest in the non NPE environment, a lower level of investment than occurs when the NPE is present.

- If $\bar{A} - \bar{C} < \bar{N}_1 \leq \bar{B} - \bar{D}$, then the NPE will invest for $\rho \in (\bar{A} - \bar{C}, \bar{N}_1]$ and one firm will also invest for ρ in this region. If $A{-}C < \bar{N}_1$, and $\rho \in (A{-}C, \bar{N}_1]$ *then in the non-NPE environment just one firm invests, whereas in the NPE environment the NPE and a firm invest.* And for $\rho \in (\bar{N}_1, \bar{B} - \bar{D}]$ one firm invests in either the NPE or non-NPE environment. If $A{-}C > \bar{N}_1$, then for $\bar{N}_1 < \rho \leq \min\{A{-}C, \bar{B} - \bar{D}\}$, since the NPE would choose not to invest, in the NPE environment both firms will invest, as they would in the non-NPE environment. And, if $A{-}C < \bar{B} - \bar{D}$, then for $\rho \in (A{-}C, \bar{B} - \bar{D}]$, in either environment just one firm invests.
- If $\bar{N}_1 \leq \bar{A} - \bar{C}$, investment is the same in both environments (two firms invest for $\rho \leq A{-}C$ and one for $A{-}C \leq \rho \leq \bar{B}{-}\bar{D}$), the investment is the same (one firm invests.)

3. Next, suppose that $\rho \in (\bar{B} - \bar{D}, B - D]$. Consider the two central cases where both $A{-}C$ and $\bar{N}_0$ lie between $\bar{B} - \bar{D}$ and $B - D$. If $A{-}C < \bar{N}_0$, then between $\bar{B} - \bar{D}$ and $A{-}C$ two firms invest in the non-NPE environment whereas in the NPE environment, the NPE alone invests. Between $A{-}C$ and $\bar{N}_0$, in the NPE environment the NPE alone invests, whereas in the non-NPE environment one firm alone invests. Finally, for ρ between $\bar{N}_0$ and $B - D$, one firm invests in either case as the NPE will never invest. Next, if $A{-}C > \bar{N}_0$ then on the region $(\bar{B} - \bar{D}, \bar{N}_0]$ in the NPE environment, the NPE alone invests whereas in the non-NPE environment both firms invest. On the region $(\bar{N}_0, A{-}C]$, both firms invest in either environment and on $(A{-}C, B{-}D]$ just one firm invests, in either environment.

4. Finally, for $\rho > B{-}D$, there is no investment in the non-NPE environment and in the NPE environment, the NPE invests only if $\bar{N}_0 > B{-}D$ and $\rho \leq \bar{N}_0$. ■

D The Linear Case: Appendix for section 5.

Equilibrium output and price with Cournot competition:

$$q_i = \frac{a - nc_i + \sum_{j \neq i} c_j}{(n+1)b}, \quad Q = \sum q_i = \frac{na - \sum c_i}{(n+1)b}, \quad P(Q) = \frac{a + \sum c_i}{(n+1)}, \quad \pi_i = \frac{1}{(n+1)^2 b}(a - nc_i + \sum_{j \neq i} c_j)^2 \tag{38}$$

PROPOSITION 6: In the linear case:

$$\psi^*(x_i) = \frac{1}{9b}[2(a - x_i)^2 + (a-1)(1-x_i)]$$

and

$$\varphi^*(z) = \begin{cases} \frac{1}{4b}(a-z)^2, & z \leq \frac{4-a}{3} \\ \frac{2}{9b}[a-3z+2][a-1], & z > \frac{4-a}{3} \end{cases}$$

Proof: Recall

$$\psi^*(x_i) = \max_{0 \leq f \leq 1-f} \{P(Q)\hat{q}_i + P(Q)\hat{q}_j - x_i(\hat{q}_i + \hat{q}_j)\} = \max_{0 \leq f \leq 1-f} \{[P(Q) - x_i][\hat{q}_i + \hat{q}_j]\}$$

Noting that quantities are determined by the cost profile $(c_i, c_j) = (x_i, x_i + f)$,

$$\begin{aligned} \hat{q}_i &= \frac{1}{3b}[a - 2x_i + (x_i + f)] = \frac{1}{3b}(a - x_i + f) \\ \hat{q}_j &= \frac{1}{3b}[a - 2(x_i + f) + x_i] = \frac{1}{3b}(a - x_i - 2f) \\ Q &= \hat{q}_i + \hat{q}_j = \frac{1}{3b}[(a - x_i + f) + (a - x_i - 2f)] = \frac{1}{3b}[2(a - x_i) - f] \end{aligned}$$

And

$$\begin{aligned} P(Q) &= \frac{1}{3}[a + x_i + (x_i + f)] = \frac{1}{3}(a + 2x_i + f) \\ P(Q) - x_i &= \frac{1}{3}(a + 2x_i + f) - x_i = \frac{1}{3}(a - x_i + f) \end{aligned}$$

So,

$$[P(Q) - x_i][\hat{q}_i + \hat{q}_j] = \frac{1}{3}(a - x_i + f)\frac{1}{3b}[2(a - x_i) - f] = \frac{1}{9b}(a - x_i + f)[2(a - x_i) - f]$$

Maximizing by choice of $0 \leq f \leq 1 - x_i$,

$$[2(a - x_i) - f] - (a - x_i + f) = 0 \Longleftrightarrow 2a - 2x_i - f - a + x_i - f = 0 \Longleftrightarrow a - x_i - 2f = 0$$

So, $\hat{f} = \min\{\frac{a-x_i}{2}, 1 - x_i\} = 1 - x_i$, since for $a \geq 2$, $\frac{a-x_i}{2} \geq 1 - \frac{1}{2}x_i \geq 1 - x_i$. Therefore,

$$\psi^*(x_i) = \frac{1}{9b}(a - x_i + \hat{f})[2(a - x_i) - \hat{f}] = \frac{1}{9b}[2(a - x_i)^2 + (a - 1)(1 - x_i)]$$

Next consider $\varphi(z)$. With $\tilde{q}_i = \tilde{q}_j = \frac{1}{3b}(a - 2(z + f) + (z + f)) = \frac{1}{3b}(a - z - f)$. $\tilde{Q} = \frac{2}{3b}(a - z - f)$. Also,

$P(\tilde{Q}) - z = a - z - b\tilde{Q} = a - z - b\frac{2}{3b}(a - z - f) = (a-z) - \frac{2}{3}(a-z) + \frac{2}{3}f = \frac{1}{3}(a-z) + \frac{2}{3}f = \frac{1}{3}[(a-z)+2f]$.

Therefore

$$\begin{aligned}
\varphi^*(z) &= \max_{0\leq f\leq 1-x_i}\{P(Q)\tilde{q}_i + P(Q)\tilde{q}_j - x_i(\tilde{q}_i+\tilde{q}_j)\} = \max_{0\leq f\leq 1-x_i}\{[P(Q)-x_i](\tilde{q}_i+\tilde{q}_j)\} \\
&= \max_{0\leq f\leq 1-x_i}\frac{1}{3}[(a-z)+2f]\frac{2}{3b}(a-z-f) = \max_{0\leq f\leq 1-x_i}\frac{2}{9b}[(a-z)+2f](a-z-f) \\
&= \frac{2}{9b}[(a-z)+2\tilde{f}](a-z-\tilde{f})
\end{aligned}$$

The maximizing choice for f, $\tilde{f}$ is a boundary solution, depending on parameter values. Differentiating $[(a-z)+2f](a-z-f)$,

$$2(a-z-f) - [(a-z)+2f] = 0 \quad \text{or} \quad (a-z)-4f = 0$$

So, $\tilde{f} = \min\{\frac{a-z}{4}, 1-z\}$. Note $\frac{a-z}{4} \leq 1-z \Leftrightarrow a-z \leq 4-4z \Leftrightarrow 3z \leq 4-a \Leftrightarrow z \leq \frac{4-a}{3}$.

$$\tilde{f} = \begin{cases} \frac{a-z}{4}, & z \leq \frac{4-a}{3} \\ 1-z, & z > \frac{4-a}{3} \end{cases}$$

$$\varphi^*(z) = \frac{2}{9b}[(a-z)+2\tilde{f}](a-z-\tilde{f})$$

For $z < \frac{4-a}{3}$,

$$\varphi^*(z) = \frac{2}{9b}[(a-z)+2\frac{a-z}{4}][a-z-(\frac{a-z}{4})] = \frac{1}{4b}(a-z)^2 = \frac{1}{9b}[\frac{9}{4}(a-z)^2]$$

For $z > \frac{4-a}{3}$,

$$\varphi^*(z) = \frac{2}{9b}[(a-z)+2(1-z)][a-z-(1-z)] = \frac{2}{9b}[a-3z+2][a-1]$$

$$\varphi^*(z) = \begin{cases} \frac{1}{9b}\frac{9}{4}(a-z)^2, & z \leq \frac{4-a}{3} \\ \frac{1}{9b}2[a-3z+2][a-1], & z > \frac{4-a}{3} \end{cases} \qquad \blacksquare$$

Corollary 1: For all y, $\psi^*(y) - \varphi^*(y) \leq 0$

Proof: Comparing $\psi^*(y)$ and $\varphi^*(y)$, for $z \leq \frac{4-a}{3}$:

$$\psi^*(y) - \varphi^*(y) \quad = \quad \frac{1}{9b}[2(a-y)^2 + (a-1)(1-y)] - \frac{1}{4b}(a-y)^2 \;=\; -\frac{1}{36b}(a+y-2)^2$$

And for $z \geq \frac{4-a}{3}$:

$$\psi^*(y) - \varphi^*(y) \quad = \quad \frac{1}{9b}\{[2(a-y)^2 + (a-1)(1-y)] - 2[a-3y+2][a-1]\} \;=\; \frac{1}{9b}(y-1)(2y+a-3)$$

Thus, for all y, $\psi^*(y) - \varphi^*(y) \leq 0$.

$$\varphi^*(y) - \psi^*(y) = \begin{cases} \frac{1}{36b}(a+y-2)^2, & y \leq \frac{4-a}{3} \\ \frac{1}{9b}(1-y)(2y+a-3), & y \geq \frac{4-a}{3} \end{cases}$$

■

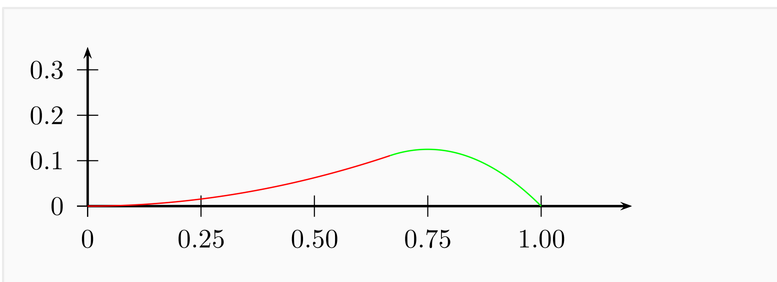


Figure 12: The function $\varphi^* - \psi^*$, $b = \frac{1}{9}$, $a = 2$.

Recall that $\psi(x) = \psi^*(x) - \pi^j(x,1)$ and $\varphi(x) = \varphi^*(x) - \pi^i(1,1) - \pi^j(1,1)$. If the firm establishes a NPE for licensing, the license revenue is $\varphi(x)$ and in addition the firm operation yields $\pi^i(1,1)$ after payment of all license fees. Therefore the two alternative revenues are: $\psi(x) = \psi^*(x) - \pi^j(x,1)$ and $\varphi(x) + \pi^i(1,1) = \varphi^*(x) - \pi^j(1,1)$. Establishing an NPE is preferable if and only if $\psi(x) \geq \varphi(x) + \pi^i(1,1)$. Or equivalently, if $\varphi^*(x) - \psi^*(x) \geq \pi^j(1,1) - \pi^j(x,1)$.

Proposition 7: For all x,

$$\varphi^*(x) - \psi^*(x) \leq \pi^j(1,1) - \pi^j(x,1)$$

Consequently, direct licensing by the firm is preferable.

PROOF: Considering the term

$$\pi^j(1,1) - \pi^j(x,1) \;=\; \frac{1}{9b}(a-1)^2 - \frac{1}{9b}(a-2+x)^2 \;=\; \frac{1}{9b}[2a-3+x](1-x)$$

Recall,

$$\varphi^*(y) - \psi^*(y) = \begin{cases} \frac{1}{36b}(a+y-2)^2, & y \leq \frac{4-a}{3} \\ \frac{1}{9b}(1-y)(2y+a-3), & y \geq \frac{4-a}{3} \end{cases}$$

There are two cases to consider: $x \leq \frac{4-a}{3}$ and $x \geq \frac{4-a}{3}$.

First consider the case where $x \geq \frac{4-a}{3}$. Then it's necessary to show that

$$\frac{1}{9b}(1-x)(2x+a-3) \;\leq\; \frac{1}{9b}(2a-3+x)(1-x) \quad \text{or} \quad x < a$$

Next, consider the case where $x \leq \frac{4-a}{3}$. This gives,

$$\begin{aligned} \frac{1}{36b}(a+x-2)^2 &\leq \frac{1}{9b}(2a-3+x)(1-x) \\ (a-1)^2 &\leq 10(a-1)(1-x) - 5(1-x)^2 \end{aligned}$$

The RHS is decreasing in x (derivative is $-10(a-1) + 10(1-x) = 10(-(a-1) + (1-x)) \leq 0$), so its smallest values occurs at $x = \frac{4-a}{3}$ which gives $1 - x = 1 - \frac{4-a}{3} = \frac{1}{3}[3 - (4-a)] = \frac{1}{3}(a-1)$. So,

$$\begin{aligned} 10(a-1)(1-x) - 5(1-x)^2 &= [10(a-1) - 5(1-x)](1-x) = [10(a-1) - \frac{5}{3}(a-1)]\frac{1}{3}(a-1) \\ &= \frac{1}{9}25(a-1)^2 \end{aligned}$$

Since $(a-1)^2 \leq \frac{25}{9}(a-1)^2$, this completes the proof. ■

D.1 Comparisons of the various functions

$\underline{\bar{A} - \bar{C}}$:

From equation (31)

$$A(z) - C(z) \;=\; \int_0^z \int_{x_i}^1 \psi(x_i) dF_j dF_i - \int_0^z \int_0^{x_i} \pi^j(x_i, 1) dF_j dF_i - \int_z^1 \int_0^z \pi^i(1,1) dF_j dF_i \tag{39}$$

$$= \int_0^z \int_{x_i}^1 \psi^*(x_i)dF_jdF_i - \int_0^z \int_0^1 \pi^j(x_i,1)dF_jdF_i - \int_z^1 \int_0^z \pi^i(1,1)dF_jdF_i \qquad (40)$$

Collecting terms:

$$\psi^*(x_i) = \frac{1}{9b}[2(a-x_i)^2 + (a-1)(1-x_i)]; \qquad \pi^j(x_i,1) = \frac{1}{9b}(a-2+x_i)^2; \qquad \pi^i(1,1) = \frac{1}{9b}(a-1)^2$$

Integrating:

$$\begin{aligned}
\int_0^z \int_{x_i}^1 \psi^*(x_i)dF(x_j)dF(x_i) &= (\frac{1}{9b})\Big[-\frac{1}{2}z^4 + \frac{1}{3}(5a+1)z^3 + \frac{1}{2}(-2a^2-6a+2)z^2 + 2a^2z + az - z)\Big] \\
\int_0^z \int_0^1 \pi^j(x_i,1)dF(x_j)dF(x_i) &= (\frac{1}{9b})\Big[\frac{1}{3}(a-2+z)^3 - \frac{1}{3}(a-2)^3\Big] \\
\int_z^1 \int_0^z \pi^i(1,1)dF(x_j)dF(x_i) &= \frac{1}{9b}(a-1)^2 z(1-z)
\end{aligned}$$

Then,

$$A(z) - C(z) = \frac{1}{9b}\Big[\frac{5}{3}z^3a - 6z^2a - \frac{1}{2}z^4 + 7az + 4z^2 - 6z\Big]$$

and so,

$$\bar{A} - \bar{C} = \frac{1}{9b}\Big[-\frac{53}{30} + \frac{23}{12}a\Big]$$

<u>$\bar{B} - \bar{D}$</u>:

From equation (32)

$$\begin{aligned}
B(z) - D(z) &= \int_0^z \int_0^1 \psi(x_i)dF(x_j)dF(x_i) - \int_0^z \int_0^1 \pi^i(1,1)dF_jdF_i \qquad (41) \\
&= \int_0^z \int_0^1 \psi^*(x_i)dF_jdF_i - \int_0^z \int_0^1 \pi^j(x_i,1)dF_jdF_i - \int_0^z \int_0^1 \pi^i(1,1)dF_jdF_i \qquad (42)
\end{aligned}$$

$$\begin{aligned}
\int_0^z \int_0^1 \psi^*(x_i)dF_jdF_i &= \frac{1}{9b}\Big[\frac{2}{3}z^3 + \frac{1}{2}(-5a+1))z^2 + 2a^2z + az - z\Big] \\
\int_0^z \int_0^1 \pi^j(x_i,1)dF_jdF_i &= (\frac{1}{9b})\Big[\frac{1}{3}(a-2+z)^3 - \frac{1}{3}(a-2)^3\Big] \\
\int_0^z \int_0^1 \pi^i(1,1)dF_jdF_i &= (\frac{1}{9b})\Big[(a-1)^2 z\Big]
\end{aligned}$$

$$B(z)-D(z)=\frac{1}{9b}\Big[\frac{1}{3}z^3-\frac{7}{2}z^2a+\frac{5}{2}z^2+7az-6z\Big] \tag{43}$$

and therefore,

$$\bar{B}-\bar{D}=\frac{1}{9b}[-\frac{25}{12}+\frac{7}{3}a] \tag{44}$$

Turning to the NPE, $\varphi(z)=\varphi^*(z)-\pi^i(1,1)-\pi^j(1,1)=\varphi^*(z)-2\pi^i(1,1)$.

$$\varphi^*(z)=\begin{cases}\frac{1}{9b}\frac{9}{4}(a-z)^2, & z\le\frac{4-a}{3}\\ \frac{1}{9b}2[a-3z+2][a-1], & z>\frac{4-a}{3}\end{cases}$$

$\underline{\bar{N}_2}$:

$$\begin{aligned}N_2(z) &= \varphi(z)[1-F(z)]^2=\varphi^*(z)[1-F(z)]^2-\pi^i(1,1)[1-F(z)]^2\\ &= \varphi^*(z)[1-z]^2-2\pi^i(1,1)[1-z]^2\end{aligned}$$

$$\bar{N}_2 = \int_0^1\varphi^*(z)[1-z]^2dz-2\frac{1}{9b}(a-1)^2\int_0^1[1-z]^2dz$$

$$(9b)\cdot\int_0^1\varphi^*(z)[1-z]^2dz = \begin{cases}\frac{9}{4}\int_0^{\frac{4-a}{3}}(a-z)^2[1-z]^2dz+2\int_{\frac{4-a}{3}}^1[a-3z+2][a-1][1-z]^2dz, & a\le 4\\ 2\int_0^1[a-3z+2][a-1][1-z]^2dz, & a>4\end{cases}$$

$$(9b)\cdot\int_0^1\varphi^*(z)[1-z]^2dz = \begin{cases}\frac{61}{810}-\frac{61}{162}a+\frac{61}{81}a^2-\frac{1}{324}a^3+\frac{1}{648}a^4-\frac{1}{3240}a^5, & a\le 4\\ -\frac{5}{6}+\frac{1}{6}a+\frac{2}{3}a^2, & a>4\end{cases}$$

$$(9b)\cdot\int_0^1 2\pi^i(1,1)[1-z]^2dz = \int_0^1 2(a-1)^2[1-z]^2dz=\frac{2}{3}(a-1)^2$$

$$(9b)\cdot\{\int_0^1 \varphi^*(z)[1-z]^2dz - \int_0^1 2\pi^i(1,1)[1-z]^2dz\} = \begin{cases} -\frac{479}{810}+\frac{155}{162}a+\frac{7}{81}a^2-\frac{1}{324}a^3+\frac{1}{648}a^4-\frac{1}{3240}a^5, & a\le 4 \\ \frac{3}{2}a-\frac{3}{2}, & a>4 \end{cases}$$

$$\bar{N}_2 = \begin{cases} \frac{1}{9b}[-\frac{479}{810}+\frac{155}{162}a+\frac{7}{81}a^2-\frac{1}{324}a^3+\frac{1}{648}a^4-\frac{1}{3240}a^5], & a\le 4 \\ \frac{1}{9b}[\frac{3}{2}a-\frac{3}{2}], & a>4 \end{cases}$$

$\underline{\bar{N}_1}$:

$$\begin{aligned} N_2(z) &= \varphi(z)[1-F(z)]^2 = \varphi^*(z)[1-F(z)]^2 - \pi^i(1,1)[1-F(z)]^2 \\ &= \varphi^*(z)[1-z]^2 - 2\pi^i(1,1)[1-z]^2 \end{aligned}$$

$$\begin{aligned} \bar{N}_1 &= \int_0^1 \varphi^*(z)[1-z]dz - 2\pi^i(1,1)\int_0^1 [1-z]dz \\ &= \int_0^1 \varphi^*(z)[1-z]dz - 2\frac{1}{9b}(a-1)^2\int_0^1 [1-z]dz \end{aligned}$$

$$(9b)\cdot\int_0^1 \varphi^*(z)[1-z]dz = \begin{cases} \frac{5}{27}+\frac{1}{108}a^3-\frac{1}{432}a^4-\frac{20}{27}a+\frac{10}{9}a^2, & a\le 4 \\ a^2-1, & a>4 \end{cases}$$

$$(9b)\cdot\{\int_0^1 \varphi^*(z)[1-z]dz - \int_0^1 2\pi^i(1,1)[1-z]dz\} = \begin{cases} -\frac{22}{27}+\frac{1}{108}a^3-\frac{1}{432}a^4+\frac{34}{27}a+\frac{1}{9}a^2, & a\le 4 \\ 2a-2, & a>4 \end{cases}$$

$$\bar{N}_1 = \begin{cases} \frac{1}{9b}[-\frac{22}{27}+\frac{34}{27}a+\frac{1}{9}a^2+\frac{1}{108}a^3-\frac{1}{432}a^4], & a\le 4 \\ \frac{1}{9b}[2a-2], & a>4 \end{cases}$$

$$N_0(z) = \varphi(z) = \varphi^*(z) - \pi^i(1,1) - \pi^j(1,1) = \varphi^*(z) - 2\pi^i(1,1)$$

$$\bar{N}_0 = \int_0^1 \varphi^*(z)dz - 2\frac{1}{9b}(a-1)^2\int_0^1 dz$$

$$(9b)\cdot\int_0^1 \varphi^*(z)dz = \begin{cases} \frac{7}{9} - \frac{7}{3}a + \frac{7}{3}a^2 - \frac{1}{36}a^3, & a \leq 4 \\ 2a^2 - a - 1, & a > 4 \end{cases}$$

$$(9b)\cdot\int_0^1 2\pi^i(1,1)dz = 2\int_0^1 (a-1)^2 dz = 2(a-1)^2$$

$$(9b)\cdot\{\int_0^1 \varphi^*(z)dz - \int_0^1 2\pi^i(1,1)dz\} = \begin{cases} -\frac{11}{9} + \frac{5}{3}a + \frac{1}{3}a^2 - \frac{1}{36}a^3, & a \leq 4 \\ 3a-3, & a \geq 4 \end{cases}$$

$$\bar{N}_0 = \begin{cases} \frac{1}{9b}[-\frac{11}{9} + \frac{5}{3}a + \frac{1}{3}a^2 - \frac{1}{36}a^3], & a \leq 4 \\ \frac{1}{9b}[3a-3], & a > 4 \end{cases}$$

$\underline{A - C}$:

From equation (34)

$$\begin{aligned} A - C &= \int_0^1\int_{x_i}^1 \psi^*(x_i)dF(x_j)dF(x_i) - \int_0^1 \pi^i(1,x_j)dF(x_j) \\ &= \int_0^1\int_{x_i}^1 \frac{1}{9b}[2(a-x_i)^2 + (a-1)(1-x_i)]dx_j dx_i - \frac{1}{9b}(a-2+x_j)dx_j \\ &= \frac{1}{9b}[-\frac{1}{6} - \frac{1}{3}a + a^2] - \frac{1}{9b}[a^2 - 3a + \frac{7}{3}] \\ &= \frac{1}{9b}[-\frac{5}{2} + \frac{8}{3}a] \end{aligned}$$

$\underline{B - D}$:

From equation (36)

$$\begin{aligned} B - D &= \int_0^1\int_0^1 \psi^*(x_i)dF_j dF_i - \int_0^1\int_0^1 \pi^i(1,1)dF_j dF_i \\ &= \int_0^1\int_0^1 \psi^*(x_i)dF_j dF_i - \int_0^1\int_0^1 \pi^j(x_i,1)dF_j dF_i - \pi^i(1,1) \end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \int_0^1 \frac{1}{9b}[2(a-x_i)^2 + (a-1)(1-x_i)]dx_j dx_i - \int_0^1 \int_0^1 \frac{1}{9b}(a-2+x_i)^2 dx_j dx_i - \frac{1}{9b}(a-1)^2 \\
&= \int_0^1 \int_0^1 \frac{1}{9b}[2(a-x_i)^2 + (a-1)(1-x_i)]dx_j dx_i - \int_0^1 \int_0^1 \frac{1}{9b}(a-2+x_i)^2 dx_j dx_i - \frac{1}{9b}(a-1)^2 \\
&= \frac{1}{9b}[\frac{1}{6} - \frac{3}{2}a + 2a^2] - \frac{1}{9b}[a^2 - 3a + \frac{7}{3}] - \frac{1}{9b}(a-1)^2 \\
&= \frac{1}{9b}[-\frac{19}{6} + \frac{7}{2}a]
\end{aligned}
$$

Figure (13) graphs the various functions as a changes (the parameter b is in all cases a scaling factor and is set to $\frac{1}{9}$).

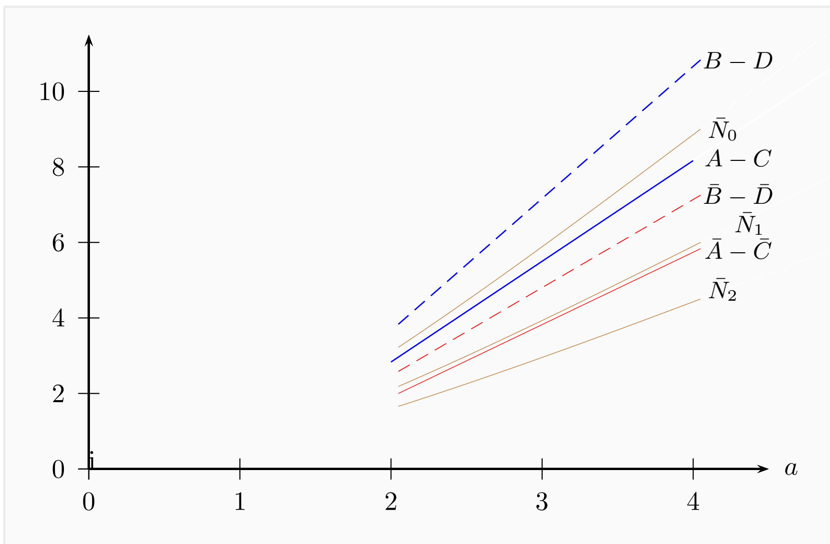


Figure 13: Payoffs ($b = \frac{1}{9}$)

The case with three firms.

Proposition 8: Suppose that three firms supply the market with demand $P = a - bQ$. Then

$A_3 - C_3$	$B_3 - D_3$	$F_3 - E_3$
$\frac{1}{480b}(-112 + 115a)$	$\frac{1}{96b}(-29 + 30a)$	$\frac{1}{48b}(-20 + 21a)$

Proof: In the linear case, with cost structure $(x_i, x_i + f, x_i + f)$, $q_i = \frac{1}{4b}(a - x_i + 2f)$ and $q_j = q_k = \frac{1}{4b}(a - x_i - 2f)$ so that total output at this cost structure is $Q = q_i + 2q_j = \frac{1}{4b}(3a - 3x_i - 2f)$ and price $P(Q) = a - bQ = \frac{1}{4}(a + 3x_i + 2f)$. In this case, $\psi_3^*(x_i) = (P(Q) - x_i)Q$. With $P(Q) - x_i = \frac{1}{4}(a - x_i + 2f)$, $\psi_3^*(x_i) = \max_{f \leq 1 - x_i} \frac{1}{4}(a - x_i + 2f)\frac{1}{4b}(3a - 3x_i - 2f)$. Differentiating with respect to f gives

$\frac{1}{16b}(2a-2x_i-4f)$, which has solution $f=\frac{a-x_i}{2} \geq \frac{2-2x_i}{2}=1-x_i$ so the optimal value of f is constrained by $f \leq 1-x_i$ and so $f=1-x_i$. At this value of f, $q_i=\frac{1}{4b}(a-3x_i+2)$, $q_j=q_k=\frac{1}{4b}(a+x_i-2)$, $Q=q_i+2q_j=\frac{1}{4b}(3a-x_i-2)$, $P(Q)=\frac{1}{4}(a+x_i+2)$, $P(Q)-x_i=\frac{1}{4}(a-3x_i+2)$, so that $\psi_3^*(x_i)=\frac{1}{16b}(a-3x_i+2)(3a-x_i-2)$. Also, $\pi^i(x_i,1,1)=\frac{1}{16b}(a-3x_i+2)$ and $\pi^j(x_i,1,1)=\pi^k(x_i,1,1)=\frac{1}{16b}(a-2+x_i)^2$. From $\psi_3(x_i)=\psi_3^*(x_i)-\pi^j(x_i,1,1)-\pi^k(x_i,1,1)$,

$$\psi_3(x_i) = \frac{1}{16b}(a-3x_i+2)(3a-x_i-2)-2\frac{1}{16b}(a-2+x_i)^2=\frac{1}{16b}\{(a-3x_i+2)(3a-x_i-2)-2(a-2+x_i)^2\}$$

Note from equation (18), with $\hat{f}=1-x_i$, $\psi_3(x_i)=\pi^i(x_i,1,1)+\hat{f}\hat{q}_j+\hat{f}\hat{q}_j$, so that $\psi_3(x_i)=\frac{1}{16b}(a-3x_i+2)^2+\frac{1}{4b}(a+x_i-2)(1-x_i)+\frac{1}{4b}(a+x_i-2)(1-x_i)=\frac{1}{16b}(a-3x_i+2)^2+2\cdot\frac{1}{4b}(a+x_i-2)(1-x_i)=\frac{1}{16b}\{(a-3x_i+2)^2+8(a+x_i-2)(1-x_i)\}$.

From these terms, $A_3=\frac{1}{160b}(-9+5a+10a^2)$, $C_3=\frac{1}{96b}(17-20a+6a^2)$ and $A_3-C_3=\frac{1}{480b}(-112+115a)$. Similarly, $B_3=\frac{1}{32b}(-5+4a+6a^2)$ and $D_3=\frac{1}{48}(7-9a+3a^2)$ so that $B_3-D_3=\frac{1}{96b}(-29+30a)$. Finally, if i is the only firm investing, the payoff is $E_3=\frac{1}{48b}(-17+15a+3a^2)$ and if not investing, $F_3=\frac{1}{16b}(a-1)^2$, so that $E_3-F_3=\frac{1}{48b}(-20+21a)$.

A_3-C_3	B_3-D_3	F_3-E_3
$\frac{1}{480b}(-112+115a)$	$\frac{1}{96b}(-29+30a)$	$\frac{1}{48b}(-20+21a)$

Comparison of these with earlier functions gives $N_1(A)<A_3(a)-C_3(a)<\bar{B}(a)-\bar{D}(a)$, $A(a)-C(a)<B_3(a)-D_3(a)<N_0(a)$, and $B(a)-D(a)<E_3(a)-F_3(a)$. ■

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