Cost Minimization and Cost Curves
Econ 212 Lecture 12

Tianyi Wang
Queen’s University
Winter 2013
We want to model how firms make use of technologies.
Take as given the objective of firms: profit maximization.
One period problem, no risks.
Opportunity cost, sunk cost and fixed cost.
Start with Cost Minimization, for
  - itself is interesting.
  - can not max profit without minimize cost
    - Profit = Revenue - Cost = $P \times Q - C(Q)$
    - Suppost cost not minimized for a set of inputs, given $Q$, can inprove profit by reducing cost.
Cost Minimization in Long-run

- Suppose firm has access to technology that relates Labor and Capital to output.
- Ask what’s the cheapest way to produce a certain level \(Q\) of output.

\[
\min_{L, K} \quad wL + rK \\
\text{s.t.} \quad Q = Q(L, K)
\]

- Long-run (less constrained) versus Short-run (more constrained).
- Use **Isocost** lines to represent firm’s objective.
  - Combinations of inputs that yield the same cost.
Optimal Input Choices

- Graphically, start with arbitrary point (A) on Isoquant, find the Isocost that goes through A.
- Ask if can find a lower Isocost. Move to B.
- Stop when Isocost just touches Isoquant (tangency). At C.
- Read the optimal inputs.
- Be aware of the difference from consumer’s utility maximization problem.
  - we moved along BL there.
Optimality Condition

- Notice at optimum slope of Isoquant equals slope of Isocost (tangency).
- Slope of Isoquant $\Delta K / \Delta L = MRTS_{L,K} = -\frac{MP_L}{MP_K}$
- Slope of Isocost $= -\frac{w}{r}$
- Then we have the following Optimality Condition:
  - $\frac{MP_L}{MP_K} = \frac{w}{r}$
- To understand, rewrite is as:
  - $\frac{MP_L}{w} = \frac{MP_K}{r}$
- Marginal Product derived from last dollar equals.
- "bang for the buck"
Why bundles like A and B are not optimal? For instance, at A Isoquant is steeper than Isocost:

\[
- \frac{MP_L}{MP_K} \leq -\frac{w}{r}, \text{ or } \frac{MP_L}{w} \geq \frac{MP_K}{r}
\]

Last dollar on $L$ produces more output than on $K$.

How to solve Cost Minimization Problem? In 3 steps:

1. Calculate $MP_L$ and $MP_K$.
2. Use Optimality Condition to express $L$ in terms of $K$ or.
3. Sub into technology constraint to solve for $K$. Then get $L$ from step 2.
Optimality Condition (Con’t)

- Optimality Condition for Fixed Proportion technology $aL = bK$.
- Corner solution for Perfect Substitute technology. Compare slopes or "bang for the buck".
  - Compare $\frac{MP_L}{w}$ and $\frac{MP_K}{r}$.
  - Produce given output by using only the higher factor.
  - Indifferent if equal.
- Note: this is minimization instead of maximization due to the nature of the problem.
- See class notes for example.
Comparative Statics

- Want to know how optimal choices change when exogenous variable changes. We will change input prices and output level.

For Cobb-Douglas technology, we solved that:

\[ L = \left( \beta r^{\alpha} w^{\alpha} \right)^{\alpha + \beta}, \]

\[ K = \left( \beta r^{\alpha} w^{\alpha} \right)^{\beta}, \]

\[ Q_1 = \left( \beta r^{\alpha} w^{\alpha} \right)^{\beta + \alpha}, \]

\[ C = \beta w^{\beta} r^{\alpha} + \beta \alpha w^{\beta} r^{\alpha + \beta}, \]

Note: these are Conditional Input Demands, conditional on output level.

Suppose wage increases from \( w_1 \) to \( w_2 \), else being equal. Factor Demand.

Suppose output increases from \( Q_1 \) to \( Q_2 \), else being equal. Expansion Path.
Comparative Statics

- Want to know how optimal choices change when exogenous variable changes. We will change input prices and output level.
- For Cobb-Douglas technology, we solved that:

\[
L^* = \left( \frac{\beta r}{\alpha w} \right)^{\frac{\alpha}{\alpha + \beta}} Q^{\frac{1}{\alpha + \beta}}
\]

\[
K^* = \left( \frac{\beta r}{\alpha w} \right)^{-\frac{\beta}{\alpha + \beta}} Q^{\frac{1}{\alpha + \beta}}
\]

\[
C^* = Q^{\frac{1}{\alpha + \beta}} w^{\frac{\beta}{\alpha + \beta}} r^{\frac{\alpha}{\alpha + \beta}} \left[ (\frac{\alpha}{\beta})^{\frac{-\alpha}{\alpha + \beta}} + (\frac{\alpha}{\beta})^{\frac{-\beta}{\alpha + \beta}} \right]
\]
Comparative Statics

- Want to know how optimal choices change when exogenous variable changes. We will change input prices and output level.
- For Cobb-Douglas technology, we solved that:

\[
L^* = \left( \frac{\beta r}{\alpha w} \right)^{\frac{\alpha}{\alpha+\beta}} Q^{\frac{1}{\alpha+\beta}} \\
K^* = \left( \frac{\beta r}{\alpha w} \right)^{-\frac{\beta}{\alpha+\beta}} Q^{\frac{1}{\alpha+\beta}} \\
C^* = Q^{\frac{1}{\alpha+\beta}} w^{\frac{\beta}{\alpha+\beta}} r^{\frac{\alpha}{\alpha+\beta}} \left[ \left( \frac{\alpha}{\beta} \right)^{-\frac{\alpha}{\alpha+\beta}} + \left( \frac{\alpha}{\beta} \right)^{-\frac{\beta}{\alpha+\beta}} \right]
\]

- Note: these are **Conditional Input Demands**, conditional on output level.
Comparative Statics

- Want to know how optimal choices change when exogenous variable changes. We will change input prices and output level.
- For Cobb-Douglas technology, we solved that:

\[
L^* = \left( \frac{\beta r}{\alpha w} \right)^{\frac{\alpha}{\alpha + \beta}} Q^{\frac{1}{\alpha + \beta}} \\
K^* = \left( \frac{\beta r}{\alpha w} \right)^{-\frac{\beta}{\alpha + \beta}} Q^{\frac{1}{\alpha + \beta}} \\
C^* = Q^{\frac{1}{\alpha + \beta}} w^{\frac{\beta}{\alpha + \beta}} r^{\frac{\alpha}{\alpha + \beta}} \left[ \left( \frac{\alpha}{\beta} \right)^{-\frac{\alpha}{\alpha + \beta}} + \left( \frac{\alpha}{\beta} \right)^{-\frac{\beta}{\alpha + \beta}} \right]
\]

- Note: these are **Conditional Input Demands**, conditional on output level.
- Suppose wage increases from \( w_1 \) to \( w_2 \), else being equal. Factor Demand.
Comparative Statics

- Want to know how optimal choices change when exogenous variable changes. We will change input prices and output level.
- For Cobb-Douglas technology, we solved that:

\[
L^* = \left( \frac{\beta r}{\alpha w} \right)^{\frac{\alpha}{\alpha + \beta}} Q^{\frac{1}{\alpha + \beta}} \\
K^* = \left( \frac{\beta r}{\alpha w} \right)^{-\frac{\beta}{\alpha + \beta}} Q^{\frac{1}{\alpha + \beta}} \\
C^* = Q^{\frac{1}{\alpha + \beta}} w^{\frac{\beta}{\alpha + \beta}} r^{\frac{\alpha}{\alpha + \beta}} \left[ \left( \frac{\alpha}{\beta} \right)^{\frac{-\alpha}{\alpha + \beta}} + \left( \frac{\alpha}{\beta} \right)^{\frac{-\beta}{\alpha + \beta}} \right]
\]

- Note: these are **Conditional Input Demands**, conditional on output level.
- Suppose wage increases from \(w_1\) to \(w_2\), else being equal. Factor Demand.
- Suppose output increases from \(Q_1\) to \(Q_2\), else being equal. Expansion Path.
Initial wage is \( w_0 \)
Wage increases from \( w_0 \) to \( w_1 \).

Holding cost constant, isocost rotates inward.

\[ Q = Q_0 \]
Shift up new isocost until tangent to Isoquant

Q=Q_0
Labor decreases
Capital increases
Cost increases

Q=Q₀
Input Demands

- We can vary wages to get optimal labor demands. Downward sloping.
Input Demands

- We can vary wages to get optimal labor demands. Downward sloping.
- We can vary rental rates to get optimal capital demands. Downward sloping.

Price elasticity of demand

\[ \epsilon = \frac{\Delta L}{L} \frac{\Delta w}{w} = \frac{dL}{dw} \]
Input Demands

- We can vary wages to get optimal labor demands. Downward sloping.
- We can vary rental rates to get optimal capital demands. Downward sloping.
- See class notes for graphs.
Input Demands

- We can vary wages to get optimal labor demands. Downward sloping.
- We can vary rental rates to get optimal capital demands. Downward sloping.
- See class notes for graphs.
- Price elasticity of demand $\epsilon = \frac{\Delta L}{\Delta w} = \frac{dL}{dw} \frac{w}{L}$
In the short-run, capital is fixed at some predetermined level, firm's problem becomes,

\[ C_s(Q, \bar{K}) = \min_{L} wL + r\bar{K} \]

\[ s.t. \; Q = Q(L, \bar{K}) \]

Simple: constraint determines optimal amount of Labor.

Note long-run cost can be written as \( C(Q) = C_s(Q, K^*(Q)) \)

LR cost equals SR cost when capital is fixed at the optimal level.
Cost Curve in Short-run

- We derived cost function algebraically, let’s now represent it graphically.
We derived cost function algebraically, let’s now represent it graphically.

Cost function in general can be written as $C(Q, w, r)$. 

Note some costs depend on output others not.

$C(Q, K) = C_v(Q, K) + F_{\text{Varaible Fixed}}$
Cost Curve in Short-run

- We derived cost function algebraically, let’s now represent it graphically.
- Cost function in general can be written as $C(Q, w, r)$.
- Since taken prices as given, it can usually be written as $C(Q)$. 

Note some costs depend on output others not.

$C(Q, K) = C_v(Q, K) + F_V$ 

See class note for graph
Cost Curve in Short-run

- We derived cost function algebraically, let’s now represent it graphically.
- Cost function in general can be written as $C(Q, w, r)$.
- Since taken prices as given, it can usually be written as $C(Q)$.
- In the short-run capital is fixed, we can write $C_s(Q, K) = C(Q, K)$.

Note some costs depend on output others not.
Cost Curve in Short-run

- We derived cost function algebraically, let’s now represent it graphically.
- Cost function in general can be written as $C(Q, w, r)$.
- Since taken prices as given, it can usually be written as $C(Q)$.
- In the short-run capital is fixed, we can write $C_s(Q, K) = C(Q, K)$.
- Note some costs depend on output others not.
We derived cost function algebraically, let’s now represent it graphically.

Cost function in general can be written as $C(Q, w, r)$.

Since taken prices as given, it can usually be written as $C(Q)$.

In the short-run capital is fixed, we can write $C_s(Q, K) = C(Q, K)$.

Note some costs depend on output others not.

\[ C(Q, K) = C_v(Q, K) + F \]

\( C_v \) Variable \( F \) Fixed
Cost Curve in Short-run

- We derived cost function algebraically, let’s now represent it graphically.
- Cost function in general can be written as $C(Q, w, r)$.
- Since taken prices as given, it can usually be written as $C(Q)$.
- In the **short-run** capital is fixed, we can write $C_s(Q, K) = C(Q, K)$.
- Note some costs depend on output others not.
  - $C(Q, K) = \underbrace{C_v(Q, K)}_{Variable} + \underbrace{F}_{Fixed}$
- See class note for graph
Average Cost Curve

- Two characteristics of cost function, AC and MC.
Two characteristics of cost function, AC and MC.

\[ AC(Q, K) = \frac{C(Q,K)}{Q} = \frac{C_v(Q,k)}{Q} + \frac{F}{Q} = AVC(Q, K) + AFC(Q, K) \]
Two characteristics of cost function, AC and MC.

\[ AC(Q, K) = \frac{C(Q, K)}{Q} = \frac{C_v(Q, k)}{Q} + \frac{F}{Q} = AVC(Q, K) + AFC(Q, K) \]

- AFC starts at infinity and declines as Q increases.
Two characteristics of cost function, AC and MC.

\[ AC(Q, K) = \frac{C(Q, K)}{Q} = \frac{C_v(Q, k)}{Q} + \frac{F}{Q} = AVC(Q, K) + AFC(Q, K) \]

- **AFC** starts at infinity and declines as Q increases.
- **AVC** is U-shape in general, but many other shapes are possible.
Average Cost Curve

- Two characteristics of cost function, AC and MC.
- \( AC(Q, K) = \frac{C(Q,K)}{Q} = \frac{C_v(Q,k)}{Q} + \frac{F}{Q} = AVC(Q, K) + AFC(Q, K) \)
- \( AFC \) starts at infinity and declines as \( Q \) increases.
- \( AVC \) is U-shape in general, but many other shapes are possible.
- See class notes for examples.
Marginal Cost Curve

\[ MC(Q, K) = \frac{\Delta C(Q, K)}{\Delta Q} = C'(Q, K) \]
Marginal Cost Curve

- \( MC(Q, K) = \frac{\Delta C(Q, K)}{\Delta Q} = C'(Q, K) \)
- Note \( MC(Q, K) = C'(Q, K) = C'_v(Q, k) + F' = C'_v(Q, k) \)
Marginal Cost Curve

- \( MC(Q, K) = \frac{\Delta C(Q, K)}{\Delta Q} = C'(Q, K) \)
- Note \( MC(Q, K) = C'(Q, K) = C'_v(Q, k) + F' = C'_v(Q, k) \)
- Relationship b/w MC and AVC

\[ MC(Q, K) = \frac{\Delta C(Q, K)}{\Delta Q} = C'(Q, K) \]

\[ MC(Q, K) = C'(Q, K) = C'_v(Q, k) + F' = C'_v(Q, k) \]
Marginal Cost Curve

- \[ MC(Q, K) = \frac{\Delta C(Q, K)}{\Delta Q} = C'(Q, K) \]
- Note \[ MC(Q, K) = C'(Q, K) = C'_v(Q, k) + F' = C'_v(Q, k) \]
- Relationship b/w MC and AVC
  - MC of the first unit is the same as the AVC of that unit.
Marginal Cost Curve

\[ MC(Q, K) = \frac{\Delta C(Q, K)}{\Delta Q} = C'(Q, K) \]

Note \( MC(Q, K) = C'(Q, K) = C'_v(Q, k) + F' = C'_v(Q, k) \)

Relationship b/w MC and AVC

- MC of the first unit is the same as the AVC of that unit.
- When AVC decreases, MC must be less than AVC.
Marginal Cost Curve

- \( MC(Q, K) = \frac{\Delta C(Q, K)}{\Delta Q} = C'(Q, K) \)
- Note \( MC(Q, K) = C'(Q, K) = C'_v(Q, k) + F' = C'_v(Q, k) \)
- Relationship b/w MC and AVC
  - MC of the first unit is the same as the AVC of that unit.
  - When AVC decreases, MC must be less than AVC.
  - When AVC increases, MC must be greater than AVC.
Marginal Cost Curve

- \( MC(Q, K) = \frac{\Delta C(Q, K)}{\Delta Q} = C'(Q, K) \)
- Note \( MC(Q, K) = C'(Q, K) = C'_v(Q, k) + F' = C'_v(Q, k) \)
- Relationship b/w MC and AVC
  - MC of the first unit is the same as the AVC of that unit.
  - When AVC decreases, MC must be less than AVC.
  - When AVC increases, MC must be greater than AVC.
  - MC intersects AVC at the minimum of AVC.
Marginal Cost Curve

- $MC(Q, K) = \frac{\Delta C(Q, K)}{\Delta Q} = C'(Q, K)$
- Note $MC(Q, K) = C'(Q, K) = C'_v(Q, k) + F' = C'_v(Q, k)$
- Relationship b/w MC and AVC
  - MC of the first unit is the same as the AVC of that unit.
  - When AVC decreases, MC must be less than AVC.
  - When AVC increases, MC must be greater than AVC.
  - MC intersects AVC at the minimum of AVC.
- Similar relationship b/w MC and AC.
Cost curves in the Long-run

- All factors can be varied in the Long-run.

\[
C(Q) = C(Q, K) = C(Q, K(Q))
\]

Short-run and Long-run average costs have the same property,

\[
AC(Q) = AC(Q, K) = AC(Q, K(Q))
\]

short-run curves always lie above long-run curve except for one point.
Cost curves in the Long-run

- All factors can be varied in the Long-run.
- Short-run costs must exceed Long-run costs, so do average costs.

\[ C(Q) = C(Q, K) \]

- Pick a output level \( Q \) and let \( K = K(Q) \) be the optimal capital level.
- Then \( C(Q) = C(Q, K) \).

Short-run and Long-run average costs have the same property,

\[ AC(Q) = AC(Q, K) \]

- Short-run curves always lie above long-run curve except for one point.
Cost curves in the Long-run

- All factors can be varied in the Long-run.
- Short-run costs must exceed Long-run costs, so do average costs.
  - $C(Q) \leq C(Q, K)$
Cost curves in the Long-run

- All factors can be varied in the Long-run.
- Short-run costs must exceed Long-run costs, so do average costs.
  - \( C(Q) \leq C(Q, K) \)
- Pick a output level \( Q^* \) and let \( K^* = K(Q^*) \) be the optimal capital level.
Cost curves in the Long-run

- All factors can be varied in the Long-run.
- Short-run costs must exceed Long-run costs, so do average costs.
  - \( C(Q) \leq C(Q, K) \)
- Pick a output level \( Q^* \) and let \( K^* = K(Q^*) \) be the optimal capital level.
  - then \( C(Q^*) = C(Q^*, K^*) \)
Cost curves in the Long-run

- All factors can be varied in the Long-run.
- Short-run costs must exceed Long-run costs, so do average costs.
  - \( C(Q) \leq C(Q, K) \)
- Pick a output level \( Q^* \) and let \( K^* = K(Q^*) \) be the optimal capital level.
  - then \( C(Q^*) = C(Q^*, K^*) \)
- Short-run and Long-run average costs have the same property,
Cost curves in the Long-run

- All factors can be varied in the Long-run.
- Short-run costs must exceed Long-run costs, so do average costs.
  - \( C(Q) \leq C(Q, K) \)
- Pick an output level \( Q^* \) and let \( K^* = K(Q^*) \) be the optimal capital level.
  - then \( C(Q^*) = C(Q^*, K^*) \)
- Short-run and Long-run average costs have the same property,
  - \( AC(Q) \leq AC(Q, K^*) \)
Cost curves in the Long-run

- All factors can be varied in the Long-run.
- Short-run costs must exceed Long-run costs, so do average costs.
  - \( C(Q) \leq C(Q, K) \)
- Pick a output level \( Q^* \) and let \( K^* = K(Q^*) \) be the optimal capital level.
  - then \( C(Q^*) = C(Q^*, K^*) \)
- Short-run and Long-run average costs have the same property,
  - \( AC(Q) \leq AC(Q, K^*) \)
  - \( AC(Q^*) = AC(Q^*, K^*) \)
Cost curves in the Long-run

- All factors can be varied in the Long-run.
- Short-run costs must exceed Long-run costs, so do average costs.
  - \( C(Q) \leq C(Q, K) \)
- Pick a output level \( Q^\ast \) and let \( K^\ast = K(Q^\ast) \) be the optimal capital level.
  - then \( C(Q^\ast) = C(Q^\ast, K^\ast) \)
- Short-run and Long-run average costs have the same property,
  - \( AC(Q) \leq AC(Q, K^\ast) \)
  - \( AC(Q^\ast) = AC(Q^\ast, K^\ast) \)
- short-run curves always lie above long-run curve except for one point.
Figure: Envelope Theorem
Long-run marginal cost curve of any output level has to equal the short-run marginal cost curve associated with the optimal level of plant size to produce that level of output.
Long-run marginal cost curve of any output level has to equal the short-run marginal cost curve associated with the optimal level of plant size to produce that level of output.

This is known as the Envelope Theorem.
Figure: Envelope Theorem
Find the long-run cost-minimizing value of costs. Solve

$$\min_{K,L} wL + rK$$

s.t. \( Q = K^\alpha L^\beta \)

How to do? First solve \( L \) from technology constraint.

$$L = Q^{1/\beta} K^{(-\alpha/\beta)}$$

then sub. into cost function to convert the multivariate minimization problem into univariate minimization problem:

$$\min_{K,L} wQ^{1/\beta} K^{(-\alpha/\beta)} + rK$$

Then solve cost-minimizing value of \( K \) from first-order-approach, and get \( L \) from tech. constraint.
Alternative derivation of Cost function (Con’t)

- **Note** it is similar as fixing $K$ in step 1 and find the optimal $L$ for a given value of capital, then sets capital equal to the value that would have been chosen if it had not been fixed.
- This is short-run cost computed at the optimal capital level!
- So, long-run cost equals the short-run cost determined at the 'appropriate' capital stock.
Math derivation of Envelope Theorem (OPTIONAL)

- By definition the following holds,

\[ C(Q) = C(Q, K(Q)) \]

- Differentiating both side w.r.t. \( Q \)

\[ \frac{dC(Q)}{dQ} = \frac{\partial C(Q, K(Q))}{\partial Q} + \frac{\partial C(Q, K(Q))}{\partial K} \frac{\partial K}{\partial Q} \]

- Suppose for a level of \( Q^* \), \( K^* = K(Q^*) \) is the optimal capital size.

- At this point we know that, by the definition of optimality, \( \frac{\partial C(Q^*, K^*)}{\partial K} = 0 \).

- Therefore at this point,

\[ \frac{dC(Q^*)}{dQ} = \frac{\partial C(Q^*, K^*)}{\partial Q} \]

- Thus the slopes of Long-run and short-run cost curves (i.e. marginal costs) are equal.