Linear Algebra: Practice Problems

September 7, 2011

ETM 2.2 A vector in $\mathbf{R}^n$ can be *normalized* by multiplying it by the reciprocal of its norm. Show that, for any $x \in \mathbf{R}^n$ with $x \neq 0$, the norm of $x/\left|\left|x\right|\right| = 1$.

ETM 2.6 Prove that, if the k columns of $\mathbf{X}$ are linearly independent, each vector z in $span(\mathbf{X})$ can be expressed as $\mathbf{Xb}$ for one and only one k-vector $\mathbf{b}$. **Hint** : Suppose that there are two different vectors, $\mathbf{b_1}$ and $\mathbf{b_2}$, such that $\mathbf{z} = \mathbf{Xb_i}$, $i = 1, 2$, and show that this implies that the columns of $\mathbf{X}$ are linearly dependent.

ETM 2.7 Consider the vectors $\mathbf{x_1} = [1, 2, 4]'$, $\mathbf{x_2} = [2, 3, 5]'$ and $\mathbf{x_3} = [3, 6, 12]'$. What is the dimension of the subspace that these vectors span?

ETM 2.14 Let $\mathbf{X}$ be an $n \times k$ matrix of full rank. Consider the $n \times k$ matrix $\mathbf{XA}$, where $\mathbf{A}$ is a *singular* $k \times k$ matrix. Show that the columns of $\mathbf{XA}$ are linearly dependent, and that $span(\mathbf{XA}) \subset span(\mathbf{X})$.

ETM 2.9 The matrix $\mathbf{P_X} = \mathbf{X}(\mathbf{X'X})^{-\mathbf{1}}\mathbf{X'}$ is a projection *projection* matrix. It projects a vector y onto the space spanned by the columns of $\mathbf{X}$. Similarly, $\mathbf{M_X} = \mathbf{I} - \mathbf{P_X}$ is a projection matrix that project y onto the space orthogonal to column space of $\mathbf{X}$, that is onto the residuals. Prove the following properties algebraically and provide a geometric intuition:

(a) $\mathbf{P_X}\mathbf{P_X} = \mathbf{P_X}$ and $\mathbf{M_X}\mathbf{M_X} = \mathbf{M_X}$

(b) $\mathbf{P_X} + \mathbf{M_X} = \mathbf{I}$ (this is why $\mathbf{P_X}$ and $\mathbf{M_X}$ are sometimes are called complimentary projections)

(c) $\mathbf{P_X}\mathbf{M_X} = \mathbf{0}$