Optimization II: Solutions for 2. and 5.

September 8, 2011

2. Minimize the following expenditure function where u is some fixed utility level

$$\min_{x_1,x_2} p_1x_1 + p_2x_2 \text{ such that } u - [x_1^\rho + x_2^\rho]^{\frac{1}{\rho}}$$

Solution: We have the following Lagrangian:

$$\mathfrak{L}(x_1, x_2, \lambda) = p_1x_1 + p_2x_2 - \lambda([x_1^\rho + x_2^\rho]^{\frac{1}{\rho}} - u)$$

The first-order conditions are:

$$\frac{\partial \mathfrak{L}}{\partial x_1} = p_1 - \lambda(1/p)[x_1^\rho + x_2^\rho]^{\frac{1}{\rho}}(px_1^{\rho-1}) = 0$$
$$\frac{\partial \mathfrak{L}}{\partial x_2} = p_2 - \lambda(1/p)[x_1^\rho + x_2^\rho]^{\frac{1}{\rho}}(px_2^{\rho-1}) = 0$$
$$\frac{\partial \mathfrak{L}}{\partial \lambda} = [x_1^\rho + x_2^\rho]^{\frac{1}{\rho}} - u = 0$$

Re-arranging the first two equations we have:

$$p_1 = \lambda[x_1^\rho + x_2^\rho]^{\frac{1}{\rho}-1}x_1^{\rho-1}$$
$$p_2 = \lambda[x_1^\rho + x_2^\rho]^{\frac{1}{\rho}-1}x_2^{\rho-1}$$

These imply that:

$$\frac{p_1}{p_2} = \frac{x_1^{\rho-1}}{x_2^{\rho-1}}$$

Re-arranging, we have

$$x_1 = [x_2^{\rho-1}\frac{p_1}{p_2}]^{\frac{1}{\rho-1}} = x_2(\frac{p_1}{p_2})^{\frac{1}{\rho-1}}$$

Using this in the constraint we have:

$$\begin{aligned}
[x_2^\rho(\frac{p_1}{p_2})^{\frac{\rho}{\rho-1}} + x_2^\rho]^{\frac{1}{\rho}} &= u \\
\implies x_2^\rho(1 + (\frac{p_1}{p_2})^{\frac{\rho}{\rho-1}})^{\frac{1}{\rho}} &= u \\
\implies x_2^* = \frac{up_2^{\frac{1}{\rho-1}}}{\left[p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right]^{1/\rho}}
\end{aligned}$$

and so $x_1^* = \frac{up_2^{\frac{1}{\rho-1}}\left[\frac{p_1}{p_2}\right]^{\frac{1}{\rho-1}}}{\left[p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right]^{1/\rho}}$.

One can verify that the Hessian of the Lagrangian $\mathfrak{L}$ is positive definitive and that (x_1^*, x_2^*) is a minimum.

5. The production function for a firm depends on capital, K, and the number of workers, L. It is given by:

$$f(K, L) = \sqrt{\sqrt{K} + \sqrt{L}}$$

The price per unit of the product is p, the cost of capital is r, and the wage rate is w, so that profit is:

$$\pi(K, L) = p\sqrt{\sqrt{K} + \sqrt{L}} - rK - wL, \ \ (K \geq 0, L \geq 0)$$

(a) Suppose that $\pi(K, L)$ has a maximum in its domain, and find the maximum point. What is the maximum when $p = 32\sqrt{2}$ and $w = r = 1$?

(b) Suppose now that the firm becomes worker-controlled, and seeks to maximize value added per worker, that is $[\pi(K, L) + wL]/L$. If we let $k = K/L$, explain why the value added per worker is

$$h(L, k) = p\left[\sqrt{1 + \sqrt{k}}\right] L^{-3/4} - rk$$

(c) Let $p = 32\sqrt{2}, r = 1$, and suppose that the corresponding function $h(L, k)$ has a maximum in the domain A consisting of all (L, k) with $L \geq 16$ and $k > 0$. Call the maximum point $(\bar{L}, \bar{k})$. Find $\bar{L}$ and show that $\bar{k} = 1$. Find the maximum value of h.

Solution: (a) We have $\Pi(K, L) = p\sqrt{\sqrt{K} + \sqrt{L}} - rK - wL$. The necessary first-order conditions for a maximum are:

$$\frac{\partial \Pi}{\partial K} = p(\frac{1}{2})(\sqrt{K} + \sqrt{L})^{-1/2} \cdot \frac{1}{2}K^{-1/2} - r = 0$$
$$\frac{\partial \Pi}{\partial K} = p(\frac{1}{2})(\sqrt{K} + \sqrt{L})^{-1/2} \cdot \frac{1}{2}L^{-1/2} - w = 0$$

From these we have:

$$\frac{p/4}{\sqrt{\sqrt{K} + \sqrt{L}}} \cdot \frac{1}{\sqrt{K}} = r \qquad (1)$$
$$\frac{p/4}{\sqrt{\sqrt{K} + \sqrt{L}}} \cdot \frac{1}{\sqrt{L}} = w \qquad (2)$$

Then, re-arranging we have:

$$L = K\frac{r^2}{w^2}$$

Now, using this result in either (1) or (2) we have after some simplification:

$$K = \left[\frac{p^2}{16(1 + r/w)r^2}\right]^{2/3}$$
$$L = \left[\frac{p^2}{16(1 + r/w)r^2}\right]^{2/3} \frac{r^2}{w^2}$$

Then, when $p = 32\sqrt{2}, w = r = 1$,

$$K = \left[\frac{64/2}{16(2)}\right]^{2/3} = 1^{2/3} = 1$$
$$L = k \cdot \frac{r^2}{w^2} = 1$$

(b) The value added per worker is simply:

$$\begin{aligned} h(L,k) = \frac{\Pi(K,L) + wL}{L} &= \frac{p\sqrt{\sqrt{K}+\sqrt{L}} - rK - wL + wL}{L} \\ &= \frac{p\sqrt{\sqrt{K}+\sqrt{L}} - rK}{L} \\ &= p\frac{\sqrt{\sqrt{K}+\sqrt{L}}}{L} - rk \\ &= p\frac{\sqrt{\sqrt{L}(\sqrt{K/L}+1)}}{L} - rk \\ &= p\frac{\sqrt{\sqrt{L}}\sqrt{\sqrt{k}+1}}{L} - rk \\ &= pL^{-3/4}\sqrt{\sqrt{k}+1} - rk \end{aligned}$$

as desired.

(c) The optimal value of L can't found using Calculus because Calculus only provides interior solutions and the solution is a corner. To see this, notice first that $h(L,k)$ is strictly decreasing in L so the choice of L that will maximize $h(L,k)$ will be the smallest possible L. Given that we have imposed $L \geq 16$, it must be that $\bar{L} = 16$. Now, to find the optimal k we simply substitute the optimal L into the objective and use Calculus to solve for $\bar{k}$. Using $p = 32\sqrt{2}, r = 1$:

$$\max_k pL^{-3/4}\sqrt{\sqrt{k}+1} - rk = 4\sqrt{2}\sqrt{\sqrt{k}+1} - k$$

Now, the first-order condition is:

$$4\sqrt{2}(1/2)(\sqrt{k}+1)^{-1/2} \cdot (1/2)k^{-1/2} - 1 = 0$$
$$\implies \frac{\sqrt{2}}{\sqrt{k}\sqrt{\sqrt{k}+1}} = 1$$

which yields a cubic equation in k that only has a root in the domain $k > 0$ at $k = 1$ so that $\bar{k} = 1$.