Optimization and Some Prob./ Stats: Practice Problems

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1. Solve the following problems. State the optimized value of the function at the solution.

   (a) $\min_{x_1,x_2} x_1^2 + x_2^2$ such that $x_1 x_2 = 1$

   (b) $\min_{x_1,x_2} x_1 x_2$ such that $x_1^2 + x_2^2 = 1$

   (c) $\min_{x_1,x_2} x_1 x_2^2$ such that $x_1^2/a^2 + x_2^2/b^2 = 1$

   (d) $\max_{x_1,x_2} x_1 + x_2$ such that $x_1^4 + x_2^4 = 1$

   (e) $\max_{x_1,x_2,x_3} x_1 x_2^2 x_3^3$ such that $x_1 + x_2 + x_3 = 1$

2. Minimize the following expenditure function where u is some fixed utility level

$$\min_{x_1, x_2} p_1 x_1 + p_2 x_2 \text{ such that } u = [x_1^\rho + x_2^\rho]^{\frac{1}{\rho}}$$

3. Mimic the procedure used in class to derive the Kuhn-Tucker conditions for the problem

$$\min_{x_1, x_2} f(x_1, x_2) \text{ s.t. } g(x_1, x_2) \le 0 \text{ and } x_1 \ge, x_2 \ge 0$$

4. Consider the following problem

$$\max_{x_1,x_2} f(x_1, x_2) \text{ such that}$$
$$a - g(x_1, x_2) \geq 0$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

Use the Kuhn-Tucker conditions to show that if we assume an interior solution and we know the constraint is binding there, the maximized value of the objective function will never decrease if the constraint is relaxed (i.e. if a rises).

5. The production function for a firm depends on capital, K, and the number of workers, L. It is given by:

$$f(K, L) = \sqrt{\sqrt{K} + \sqrt{L}}$$

The price per unit of the product is p, the cost of capital is r, and the wage rate is w, so that profit is:

$$\pi(K, L) = p\sqrt{\sqrt{K} + \sqrt{L}} - rK - wL, \quad (K \geq 0, L \geq 0)$$

(a) Suppose that $\pi(K, L)$ has a maximum in its domain, and find the maximum point. What is the maximum when $p = 32\sqrt{2}$ and $w = r = 1$?

(b) Suppose now that the firm becomes worker-controlled, and seeks to maximize value added per worker, that is $[\pi(K, L) + wL]/L$. If we let $k = K/L$, explain why the value added per worker is

$$h(L, k) = p\left[\sqrt{1 + \sqrt{k}}\right] L^{-3/4} - rk$$

(c) Let $p = 32\sqrt{2}, r = 1$, and suppose that the corresponding function $h(L, k)$ has a maximum in the domain A consisting of all (L, k) with $L \geq 16$ and $k > 0$. Call the maximum point $(\bar{L}, \bar{k})$. Find $\bar{L}$ and show that $\bar{k} = 1$. Find the maximum value of h.