Queen's University
Faculty of Arts and Sciences
Department of Economics
Graduate Methods Review Course 2009
Exit Exam
Instructions: 2 Hours

You are to answer **ALL** questions. **SHOW ALL YOUR WORK**. There are a total of 100 possible marks to be obtained and marks are indicated for each question.

1. (**10 Marks**) Sketch a few level sets for the following functions. Are the level sets convex?

   (a) $y = x_1 x_2$

   (b) $y = x_1 + x_2$

   (c) $y = min[x_1, x_2]$

   Solution: The levels sets are certainly all convex functions as is clear from the graphs.

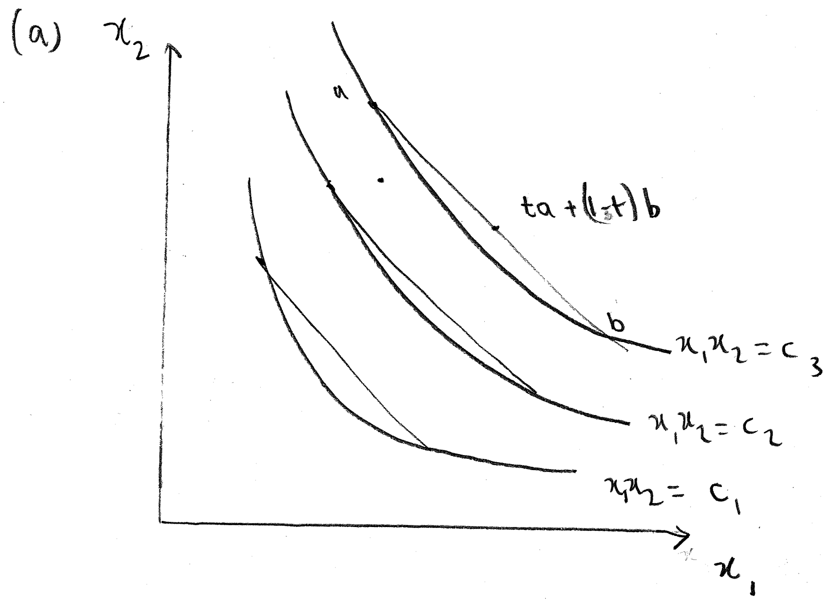

Figure 1: a.

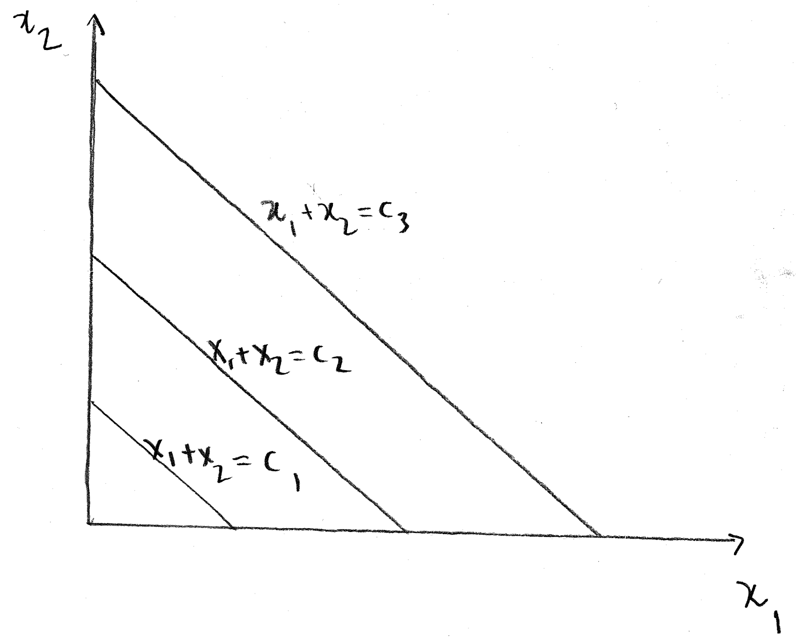

Figure 2: b.

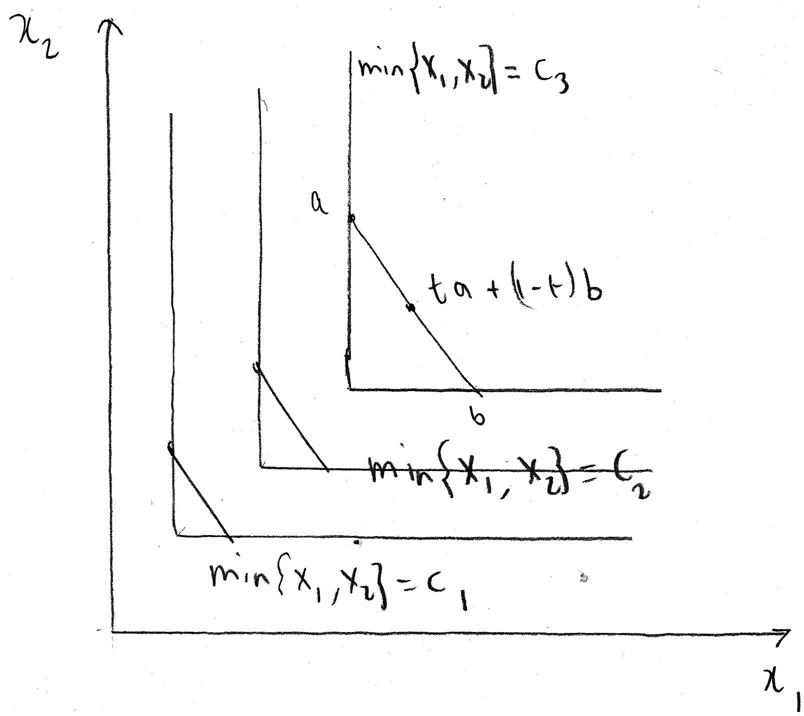

Figure 3: c.

2. (**10 Marks**) Let $f : D \to R$ be any mapping and let B be any set in the range R. Prove that

$$f^{-1}(B^c) = (f^{-1}(B))^c$$

(*If you can't prove it analytically, at least draw a convincing picture.*)

Solution:

We need to show

(a) $f^{-1}(B^c) \subset (f^{-1}(B))^c$

(b) $(f^{-1}(B))^c \subset f^{-1}(B^c)$

To see (a), note that if $x \in f^{-1}(B^c)$ then there exists a $y \in B^c$ such that $f(x) = y$. Moreover, as $y \notin B$, $f^{-1}(y) \notin f^{-1}(B) \implies f^{-1}(y) \in (f^{-1}(B))^c$.

Similarly, to see (b), note that if $x \in (f^{-1}(B))^c$ then $x \notin (f^{-1}(B))$. That is, there is no $y \in B$ such that $f(x) = y$. Hence, there must be a $y \in B^c$ s.t. $f(x) = y \implies x \in f^{-1}(B^c)$.

3. (**10 Marks**) Let $f(x_1, x_2) = (x_1 x_2)^2$. Is $f(x_1, x_2)$ concave on $\mathbb{R}^2_+$?

Solution: As $f(x_1, x_2)$ is clearly twice-differentiable on $\mathbb{R}^2_+$, it is convex iff $H(x_1, x_2)$ is positive semi-definite for all $(x_1, x_2) \in \mathbb{R}^+$.

Notice that:

$$\frac{\partial f}{\partial x_1} = 2x_1(x_2)^2$$
$$\frac{\partial f}{\partial x_2} = 2(x_1)^2 x_2$$
$$\frac{\partial^2 f}{\partial x_1^2} = 2(x_2)^2$$
$$\frac{\partial^2 f}{\partial x_2^2} = 2(x_1)^2$$
$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = 4x_1 x_2$$

The Hessian is:

$$\begin{bmatrix} 2(x_2)^2 & 4x_1x_2 \\ 4x_1x_2 & 2(x_1)^2 \end{bmatrix}$$

The principal minors are:

- $D_1 = |2(x_2)^2| = 2(x_2)^2 > 0$ for all $x_2 \in R^2_+$
- $D_2 = \begin{vmatrix} 2(x_2)^2 & 4x_1x_2 \\ 4x_1x_2 & 2(x_1)^2 \end{vmatrix} = -12(x_1x_2)^2 < 0$ for all $(x_1, x_2) \in \mathbb{R}^2_+$

Since, the principal minors alternate starting with a positive, the Hessian is NEITHER positive or negative definite, implying that $f(x_1, x_2) = (x_1x_2)^2$ is neither concave or convex $\mathbb{R}^2_+$.

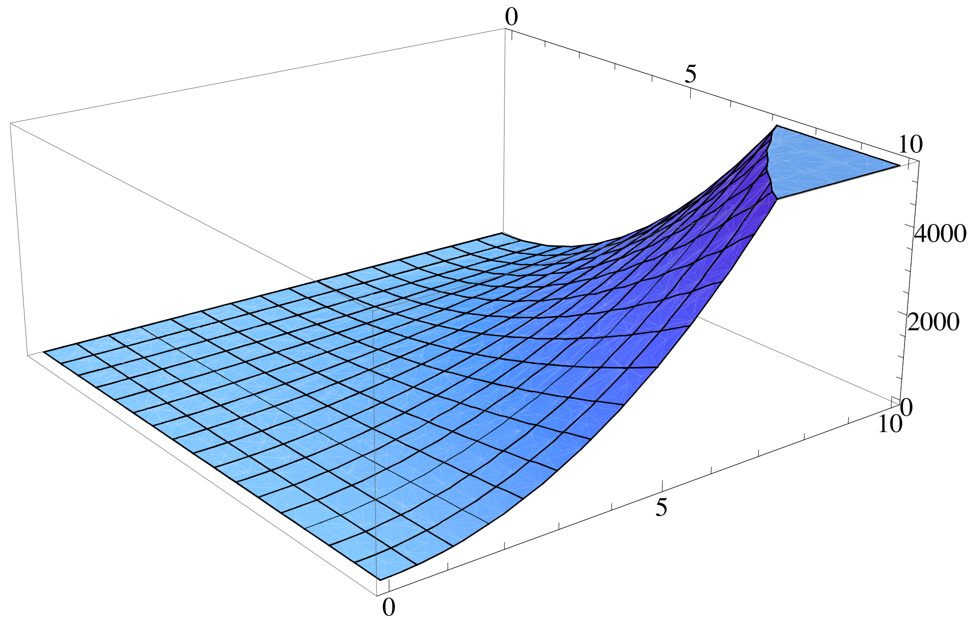

Figure 4: $f(x_1, x_2) = (x_1 x_2)^2$

4. (**10 Marks**) Consider a producer who rents machines K at r per year and hires labour L at wage w per year to produce output Q, where

$$Q = \sqrt{K} + \sqrt{L}$$

Suppose he wishes to produce a fixed quantity Q at minimum cost. Find his factor demand functions. That is, find the optimal levels of capital and labour in the following problem:

$$\min_{K,L} rK + wL \text{ s.t. } Q = \sqrt{K} + \sqrt{L}$$

Also, show that the Lagrange multiplier is given by

$$\lambda = 2wrQ/(w + r)$$

Solution:

The Lagrangian for the above problem is:

$$\mathfrak{L}(K, L, \lambda) = rK + wL + \lambda[Q - \sqrt{K} - \sqrt{L}]$$

The corresponding first-order conditions are:

$$\frac{\partial \mathfrak{L}}{\partial K} = r - \frac{1}{2}\lambda K^{-1/2} = 0 \tag{1}$$

$$\frac{\partial \mathfrak{L}}{\partial L} = w - \frac{1}{2}\lambda L^{-1/2} = 0 \tag{2}$$

$$\frac{\partial \mathfrak{L}}{\partial \lambda} = Q - \sqrt{K} - \sqrt{L} = 0 \tag{3}$$

Equating (1) and (2) we have:

$$\frac{r}{w} = \frac{k^{-1/2}}{L^{-1/2}} \tag{4}$$

Together with the constraint, we can solve for the optimal values of K, L:

$$\sqrt{K} = \frac{Q}{1 + r/w} \implies K^* = \frac{Q^2}{(1 + r/w)^2} \tag{5}$$

$$\sqrt{L} = \frac{Q \cdot r/w}{1 + r/w} \implies L^* = \frac{Q^2 (r/w)^2}{(1 + r/w)^2} \tag{6}$$

Using either of these expressions in (1) or (2) we can recover the multiplier:

$$\lambda = \frac{2rQ}{1 + r/w} = \frac{2rwQ}{w + r} \tag{7}$$

5. (**10 Marks**) There is a fixed total Y of goods at the disposal of society. There are two consumers who envy each other. If consumer 1 gets Y_1 and consumer 2 gets Y_2, their utilities are

$$U_1 = Y_1 - kY_2^2$$
$$U_2 = Y_2 - kY_1^2$$

where k is a positive constant. The allocation must satisfy $Y_1 + Y_2 \leq Y$, and maximize $U_1 + U_2$. Show that if $Y > 1/k$, the resource constraint will be slack at the optimum. Interpret the result.

Solution: We are to solve the following problem:

$$\max_{Y_1, Y_2} (Y_1 - kY_2^2) + (Y_2 - kY_1^2) \text{ s.t. } Y_1 + Y_2 \leq Y \tag{8}$$

Let $z \geq 0$ be such that

$$Y - (Y_1 + Y_2) - z = 0$$

Then solving (8) is the equivalent to solving the following problem:

$$\max_{Y_1, Y_2,, \lambda, z} \mathfrak{L}(Y_1, Y_2, \lambda, z) = (Y_1 - kY_2^2) + (Y_2 - kY_1^2) + \lambda(Y - (Y_1 + Y_2) - z) \text{ s.t. } z \geq 0$$

The necessary conditions for a solution are:

$$
\begin{aligned}
\frac{\partial \mathfrak{L}}{\partial Y_1} &= 1 - 2kY_1 - \lambda = 0 && (9)\\
\frac{\partial \mathfrak{L}}{\partial Y_2} &= 1 - 2kY_2 - \lambda = 0 && (10)\\
\frac{\partial \mathfrak{L}}{\partial \lambda} &= Y - (Y_1 + Y_2) - z = 0 && (11)\\
\frac{\partial \mathfrak{L}}{\partial z} &= -\lambda \leq 0 && (12)\\
\frac{\partial \mathfrak{L}}{\partial Y_1} z &= -\lambda z = 0 && (13)\\
z &\geq 0 && (14)
\end{aligned}
$$

Starting with (13), notice that there are two cases to consider:

(a) $\lambda > 0$**,** $z = 0$: constraint binds as $z = 0$.

Notice that these values of λ and z satisfy (12) and (14). From (9) and (10), we obtain $1 - 2kY_1 = 1 - 2kY_2 \implies Y_1 = Y_2$. Using this result in (11) together with $z = 0$, we have $Y_1 = Y_2 = Y/2$ at the optimum. Moreover, $\lambda = 1 - 2k * (1/2)Y = 1 - Y/k > 0$ as long as $Y > 1/k$.

(b) $\lambda = 0$**,** $z > 0$: constraint is slack as $z > 0$

Again, notice that these values are consistent with (12) and (14). Then, from (9) and (10) we obtain $Y_1 = Y_2 = 1/2k$. Using this result in (11) we obtain that $z = Y - 1/k$. Now, to ensure that (14) is satisfied with this value for z, we need $Y > 1/k$. Hence, the resource constraint is slack whenever this is true as it implies a strictly positive z.

(Note: technically, there is a third case to be considered: $\lambda = z = 0$ but it doesn't affect the above discussion since in this case the constraint is also binding.)

6. (**10 Marks**) If $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ are linearly independent vectors in $\mathbb{R}^m$, prove that $\mathbf{a} + \mathbf{b}$, $\mathbf{b} + \mathbf{c}$ and $\mathbf{a} + \mathbf{c}$ are also linearly independent. Is the same true for $\mathbf{a} - \mathbf{b}$, $\mathbf{b} + \mathbf{c}$ and $\mathbf{a} + \mathbf{c}$?

Solution: We need to show that $\mathbf{a} + \mathbf{b}$, $\mathbf{b} + \mathbf{c}$ and $\mathbf{a} + \mathbf{c}$ are also linearly independent. In other words, we need to show that the only solution to:

$$\alpha_1(\mathbf{a} + \mathbf{b}) + \alpha_2(\mathbf{b} + \mathbf{c}) + \alpha_3(\mathbf{a} + \mathbf{c}) = 0$$

is $\alpha_1 = \alpha_2 = \alpha_3 = 0$. To see this note that

$$
\begin{aligned}
&\alpha_1(\mathbf{a} + \mathbf{b}) + \alpha_2(\mathbf{b} + \mathbf{c}) + \alpha_3(\mathbf{a} + \mathbf{c}) = (\alpha_1 + \alpha_3)\mathbf{a} + (\alpha_1 + \alpha_2)\mathbf{b} + (\alpha_2 + \alpha_3)\mathbf{c}\\
&\implies \alpha_1 + \alpha_3 = \alpha_1 + \alpha_2 = \alpha_2 + \alpha_3 = 0
\end{aligned}
$$

as $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are linearly independent. Solving this system, we have $\alpha_2 = \alpha_3 = \alpha_1 \implies \alpha_1 = \alpha_2 = \alpha_3 = 0$. Hence, $\mathbf{a} + \mathbf{b}$, $\mathbf{b} + \mathbf{c}$ and $\mathbf{a} + \mathbf{c}$ are also linearly independent.

Applying the same technique to $\mathbf{a} - \mathbf{b}$, $\mathbf{b} + \mathbf{c}$, $\mathbf{a} + \mathbf{c}$ we find that $\alpha_1 + \alpha_3 = -\alpha_1 + \alpha_2 = \alpha_2 + \alpha_3 = 0$. Solving this system, we find that $\alpha_1 = \alpha_2 = -\alpha_3$ Here, there are there are lots of solutions. For example, $\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = -1$ ensures that

$$\alpha_1(\mathbf{a} - \mathbf{b}) + \alpha_2(\mathbf{b} + \mathbf{c}) + \alpha_3(\mathbf{a} + \mathbf{c}) = 0$$

Hence, $\mathbf{a}-\mathbf{b}$, $\mathbf{b}+\mathbf{c}$, $\mathbf{a}+\mathbf{c}$ are linearly dependent.

7. (**10 Marks**) Suppose that $\mathbf{a},\mathbf{b},\mathbf{c}\in\mathbb{R}^3$ are all different from $\mathbf{0}$, and that $\mathbf{a}\perp\mathbf{b}$, $\mathbf{b}\perp\mathbf{c}$, $\mathbf{a}\perp\mathbf{c}$. Prove that $\mathbf{a},\mathbf{b}$, and $\mathbf{c}$ are linearly independent.

   Solution: We need to show that the only solution to

$$\alpha_1\mathbf{a}+\alpha_2\mathbf{b}+\alpha_3\mathbf{c}=0 \tag{15}$$

   is $\alpha_1=\alpha_2=\alpha_3=0$. Notice that when doting the LHS with $\mathbf{a}$ we have

$$\begin{aligned}(\alpha_1\mathbf{a}+\alpha_2\mathbf{b}+\alpha_3\mathbf{c})\cdot\mathbf{a}&=\alpha_1\mathbf{a}\cdot\mathbf{a}+\alpha_2\mathbf{b}\cdot\mathbf{a}+\alpha_3\mathbf{c}\cdot\mathbf{a}\\&=\alpha_1\mathbf{a}\cdot\mathbf{a}\end{aligned}$$

   as $\mathbf{b}\cdot\mathbf{a}=\mathbf{c}\cdot\mathbf{a}=0$ as a result of orthogonality. Doting the RHS of (15) with $\mathbf{a}$ we have $\mathbf{0}\cdot\mathbf{a}=\mathbf{0}$. Hence,

$$(\alpha_1\mathbf{a}+\alpha_2\mathbf{b}+\alpha_3\mathbf{c})\cdot\mathbf{a}=\alpha_1\mathbf{a}\cdot\mathbf{a}=\mathbf{0}$$

   Furthermore, $\mathbf{a}\cdot\mathbf{a}\neq 0$ as $\mathbf{a}\cdot\mathbf{a}$ is the norm of $\mathbf{a}$ as the latter is different from $\mathbf{0}$. Therefore, α_1 must be zero. Similarly, by considering

$$\begin{aligned}&(\alpha_1\mathbf{a}+\alpha_2\mathbf{b}+\alpha_3\mathbf{c})\cdot\mathbf{b}\\ \text{and }&(\alpha_1\mathbf{a}+\alpha_2\mathbf{b}+\alpha_3\mathbf{c})\cdot\mathbf{c}\end{aligned}$$

   we can show that α_2, and α_3 are both zero. Hence, the vectors $\mathbf{a},\mathbf{b},\mathbf{c}$ are linearly independent.

8. (**10 Marks**) Let $(\mathbf{Y}_1,\mathbf{Y}_2,...,\mathbf{Y}_n)$ be a random sample of size n drawn from a population distribution $F_Y(y)$. The sample mean is defined as

$$\overline{Y}=\sum_{i=1}^{n}Y_i$$

   What is meant by the statement that the sample mean has a sampling distribution?

   Solution:

   For each sample taken from the population we can construct a sample mean. Since the observations vary across samples, the sample mean will also vary across samples. The overall variation is captured by the sampling distribution. (Plotting a cumulative relative frequency histogram of the values of the sample mean across all samples generates the sampling distribution.)

9. (**10 Marks**) Hughes Tool Company has a contract to build a piece of equipment it has never built before. The contract requires that the project be completed within 90 days. Let H denote the hypothesis that the project will be completed within 90 days, and let $\bar{H}$ denote the opposite hypothesis that the project will take longer than 90 days to complete. One can assign to H a probability $P(H)$ that represents numerically the degree of a person's belief in H.

   Now our contracted company hires a work methods engineer to determine how long it will take to complete the project, and she says that it can be completed within the specified time. This new information, called D (for data), will certainly alter our attitude toward the hypothesis H. Still, we know that the methods engineer is not always correct. After asking some questions, we determine that 80% of the projects that are completed on time have been correctly forecast as being completed on time by this engineer. That is, $P(D|H) = 0.8$. We also know that for projects that were not completed on time, the engineer forecasted completion by the deadline 10% of the time; thus $P(D|\bar{H}) = 0.10$.

Please compute a *new* probability $P(H|D)$ of the hypothesis, given the data from the engineer and given that $P(H) = 0.3$.

Hint : try to formulate this problem in terms of Bayes' Theorem.

Solution:

A simple application of Bayes' Theorem says that:

$$P(H|D) = \frac{P(D|H)P(H)}{P(D)}$$

We know that $P(D|H) = 0.8$ and that $P(H) = 0.8$. The only term we don't know is P(D). So how do we determine P(D)? Note that since the events H and $\bar{H}$ are mutually exclusive:

$$\begin{aligned} P(D) &= P(D \cap H) + P(D \cap \bar{H}) \\ &= P(D|H)P(H) + P(D|\bar{H})P(\bar{H}) \\ &= 0.8 \times 0.3 + 0.1 \times 0.7 \\ &= 0.31 \end{aligned}$$

Then,

$$\begin{aligned} P(H|D) &= \frac{P(D|H)P(H)}{P(D)} \\ &= \frac{0.8 \times 0.3}{0.31} \\ &= 0.77. \end{aligned}$$

That is, observing D increases our personal probability of H because H "explains" D better than $\bar{H}$ does: $P(D|H) > P(D|\bar{H})$.

10. (**10 Marks**) If X is a random number selected from the first 10 positive integers, what is E[X(11 - X)]? What is Var[X(11 - X)]?

    Solution: One might be tempted to approach the problem algebraically. That is, by expanding the expectation as follows:

$$\begin{aligned} E(X(11 - X)) &= E(11X) - E(X^2) \\ &= 11 \times E(X) - E(X^2) \end{aligned}$$

and then computing $E(X)$ and $E(X^2)$ for the random variable in question. However, while this works, it is tedious since you need to carry out the calculations for $E(X^2)$ and in the second part of the question, for $V(X^2)$. A somewhat less tedious approach is to simply calculate the realized values of the random variable $X(11 - X)$:

X	X(11-X)
1	10
2	18
3	24
4	28
5	30
6	30
7	28
8	24
9	18
10	10

Hence, computing the expectation we have:

$E(X(11-X) = \frac{1}{10} \times 10 + \frac{1}{10} \times 18 + \cdots + \frac{1}{10} \times 10 = 22.$

Similarly,

$Var(X(11-X) = \frac{1}{10} \times (10-22)^2 + \frac{1}{10} \times (18-22)^2 + \cdots + \frac{1}{10} \times (10-22)^2 = 52.8.$