Queen's University
Faculty of Arts and Sciences
Department of Economics
Graduate Methods Review Course 2009
Exit Exam
Instructions: 2 Hours

You are to answer **ALL** questions. **SHOW ALL YOUR WORK**. There are a total of 100 possible marks to be obtained and marks are indicated for each question.

1. (**10 Marks**) Sketch a few level sets for the following functions. Are the level sets convex?
   (a) $y = x_1 x_2$
   (b) $y = x_1 + x_2$
   (c) $y = min[x_1, x_2]$

2. (**10 Marks**) Let $f : D \to R$ be any mapping and let B be any set in the range R. Prove that

$$f^{-1}(B^c) = (f^{-1}(B))^c$$

   (*If you can't prove it analytically, at least draw a convincing picture.*)

3. (**10 Marks**) Let $f(x_1, x_2) = (x_1 x_2)^2$. Is $f(x_1, x_2)$ concave on $\mathbb{R}^2_+$?

4. (**10 Marks**) Consider a producer who rents machines K at r per year and hires labour L at wage w per year to produce output Q, where

$$Q = \sqrt{K} + \sqrt{L}$$

   Suppose he wishes to produce a fixed quantity Q at minimum cost. Find his factor demand functions. That is, find the optimal levels of capital and labour in the following problem:

$$\min_{K,L} rK + wL \text{ s.t. } Q = \sqrt{K} + \sqrt{L}$$

   Also, show that the Lagrange multiplier is given by

$$\lambda = 2wrQ/(w + r)$$

5. (**10 Marks**) There is a fixed total Y of goods at the disposal of society. There are two consumers who envy each other. If consumer 1 gets Y_1 and consumer 2 gets Y_2, their utilities are

$$U_1 = Y_1 - kY_2^2$$
$$U_2 = Y_2 - kY_1^2$$

   where k is a positive constant. The allocation must satisfy $Y_1 + Y_2 \leq Y$, and maximize $U_1 + U_2$. Show that if $Y > 1/k$, the resource constraint will be slack at the optimum. Interpret the result.

6. (**10 Marks**) If $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ are linearly independent vectors in $\mathbb{R}^m$, prove that $\mathbf{a} + \mathbf{b}$, $\mathbf{b} + \mathbf{c}$ and $\mathbf{a} + \mathbf{c}$ are also linearly independent. Is the same true for $\mathbf{a} - \mathbf{b}$, $\mathbf{b} + \mathbf{c}$ and $\mathbf{a} + \mathbf{c}$?

7. (**10 Marks**) Suppose that $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ are all different from $\mathbf{0}$, and that $\mathbf{a} \perp \mathbf{b}$, $\mathbf{b} \perp \mathbf{c}$, $\mathbf{a} \perp \mathbf{c}$. Prove that $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ are linearly independent.

8. (**10 Marks**) Let $(\mathbf{Y}_1, \mathbf{Y}_2, ..., \mathbf{Y}_n)$ be a random sample of size n drawn from a population distribution $F_Y(y)$. The sample mean is defined as

$$\overline{Y} = \sum_{i=1}^{n} Y_i$$

   What is meant by the statement that the sample mean has a sampling distribution?

9. (**10 Marks**) Hughes Tool Company has a contract to build a piece of equipment it has never built before. The contract requires that the project be completed within 90 days. Let H denote the hypothesis that the project will be completed within 90 days, and let $\bar{H}$ denote the opposite hypothesis that the project will take longer than 90 days to complete. One can assign to H a probability $P(H)$ that represents numerically the degree of a person's belief in H.

   Now our contracted company hires a work methods engineer to determine how long it will take to complete the project, and she says that it can be completed within the specified time. This new information, called D (for data), will certainly alter our attitude toward the hypothesis H. Still, we know that the methods engineer is not always correct. After asking some questions, we determine that 80% of the projects that are completed on time have been correctly forecast as being completed on time by this engineer. That is, $P(D|H) = 0.8$. We also know that for projects that were not completed on time, the engineer forecasted completion by the deadline 10% of the time; thus $P(D|\bar{H}) = 0.10$.

   Please compute a *new* probability $P(H|D)$ of the hypothesis, given the data from the engineer and given that $P(H) = 0.3$.

   Hint : try to formulate this problem in terms of Bayes' Theorem.

10. (**10 Marks**) If X is a random number selected from the first 10 positive integers, what is E[X(11 - X)]? What is Var[X(11 - X)]?