Queen's University
Faculty of Arts and Sciences
Department of Economics
Graduate Methods Review Course 2010
Exit Exam
Instructions: 2 Hours

You are to answer **ALL** questions. **SHOW ALL YOUR WORK**. There are a total of 100 possible marks to be obtained and marks are indicated for each question.

1. (**10 Marks**) Let A, B be convex sets.

   (a) Prove that $A \times B = \{(a, b)|a \in A, b \in B\}$, that is the set of pairs with the first element from A and the second element from B, is convex.

   (b) Let $A, B \subset \mathbb{R}$. Now assume that B is NOT convex. Show using a counterexample that then $A \times B$ is not convex.

   Solution:

   (a) Let (a_1, b_1) and (a_2, b_2) be any two points in $A \times B$. Then the convex combination of these points is
   $$p = (ta_1 + (1 - t)a_2, tb_1 + (1 - t)b_2)$$
   for all $t \in [0, 1]$. Now, the first element is in A because A is convex and the second element is in B because B is convex. So $p \in A \times B$. Hence, $A \times B$ is convex.

   (b) Let $A = [0, 1]$ and $B = [0, 1/3] \cup [2/3, 1]$ then $A \times B = [0, 1] \times [0, 1/3] \cup [0, 1] \times [2/3, 1]$. Then $(1/4, 1/4)$ and $(3/4, 3/4)$ are in $A \times B$ but their convex combination with $t = 1/2$ is not: i.e. $(1/2, 1/2) \notin A \times B$.

2. (**10 Marks**) Let $g(\mu) = A(1 + \mu)^{\frac{a}{1+b}} - 1$. Find:

   (a) $\frac{dg}{d\mu}$.

   (b) a linear approximation of $g(\mu)$ around the point 0 (i.e. the first-order Taylor series)

   For $a = 1, b = 7$, use the approximation AND $g(\mu)$ to compute $g(0.1)$ and find the error of your approximation.

   Solution:

   (a) $\frac{dg}{d\mu} = A\left(\frac{a}{1+b}\right)(1 + \mu)^{\frac{a}{1+b}-1}$

   (b) The first-order approximation to $g(\mu)$ around $\mu = 0$ is
   $$\begin{aligned} g(\mu) &\approx g(0) + \frac{dg}{d\mu}(0)(\mu - 0) \\ &= A(1 + 0)^{\frac{a}{1+b}} - 1 + [A\left(\frac{a}{1 + b}\right)(1 + 0)^{\frac{a}{1+b}-1}](\mu) \\ &= A - 1 + \left(\frac{Aa}{1 + b}\right)\mu \\ &= \mu/8 \end{aligned}$$
   using the values $A = 1, a = 1, b = 7$. Hence, $g(0.1) \approx 0.1/8 = 0.0125$. The exact value of $g(0.1)$ is $0.011985\ldots$ so our crude approximation isn't so bad.

3. (**10 Marks**) Don't order "chicken" in K-Town.
If the total number T, of birds that get turned into fresh meat in a week, by Y wind turbines on Wolf Island is $T = a(bY + c)^p + kY$, where a, b, and c are positive constants, then the average kill rate is:

$$\overline{T}(Y) = \frac{T}{Y} = a\frac{(bY + c)^p}{Y} + k \qquad (p > 1)$$

Find the value of Y that maximizes the average kill rate.

Solution: We are trying to find

$$\max_Y \overline{T}(Y)$$

A necessary condition for a maximum is that $\overline{T}'(Y) = 0$. Taking the derivative and setting it to zero we have:

$$\begin{aligned}
\frac{ap(bY + c)^{p-1}}{Y} \cdot b + \left(\frac{-1}{Y^2}\right) a(bY + c)^p = 0 \\
\implies \frac{ap(bY + c)^{p-1}}{Y} \cdot b = \left(\frac{-1}{Y^2}\right) a(bY + c)^p \\
\implies \frac{pY}{a} = (bY + c) \\
\implies Y = \frac{c}{p/a - b}
\end{aligned}$$

where I have ignored the solution $Y = 0$ because $\overline{T}$ is not defined for $Y = 0$. BUT THIS IS NOT A MAXIMUM, IT IS A MINIMUM. WHY? Intuitively, for $p > 1$, the objective is strictly increasing and convex in Y for $Y > 1$ and so in fact there is no maximum, more Y is always increases the kill rate.

4. (**10 Marks**) Prove that if f and g are both concave, then

$$h(x) = \min\{f(x), g(x)\}$$

is concave. Illustrate. (Note that for each given x, $h(x)$ is the smaller of the two numbers $f(x)$ and $g(x)$.)

Solution: We are going to do what comes naturally here. Let x_1 and x_2 be any two points. For any $t \in [0, 1]$ denote the convex combination of x_1 and x_2 as $x_t = tx_1 + (1 - t)x_2$. Then we are to show that:

$$h(x_t) \geq th(x_1) + (1 - t)h(x_2)$$

Assume that $h(x_t) = f(x_t)$ (one of $f(x_t)$ or $g(x_t)$ has to be the minimum, I picked $f(x_t)$). Then, notice that:

$$\begin{aligned}
h(x_t) &= f(x_t) \\
&\geq tf(x_1) + (1 - t)f(x_2) \qquad \text{because } f \text{ is concave} \\
&\geq th(x_1) + (1 - t)h(x_2)
\end{aligned}$$

as $f(x_1) \geq h(x_1)$ and $f(x_2) \geq h(x_2)$ by noting the definition of $h(x)$. Hence $h(x)$ is concave. Now, if $h(x_t) = g(x_t)$ the exact same argument can be applied to show that $h(x)$ is concave.

5. (**15 Marks**) Channelling your inner Keanu Reeves.

(a) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

i. Show that $A^2 = (a+d)A - (ad-bc)I_2$

ii. Use part (i) to show that $A^3 = 0$ implies that $A^2 = 0$. (Hint: Multiply the equality in part (a) by A, and use the equality $A^3 = 0$ to derive an equation, which you should then multiply by A once again.)

iii. Give an example of a matrix A such that $A^2 = A^3 = 0$, but $A \neq 0$.

(b) Suppose P and Q are $n \times n$ matrices such that $PQ = Q^2 P$. Prove that $(PQ)^2 = Q^6 P^2$.

Solution:

(a) i. LHS $= A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{pmatrix}$. RHS $= (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} -$ $(ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$

ii. Multiplying $A^2 = (a+d)A - (ad-bc)I_2$ by A once we have:

$$\begin{aligned} & A \cdot A^2 = A \cdot (a+d)A - A \cdot (ad-bc)I_2 \\ & \implies A^3 = (a+d)A^2 - (ad-bc)A \quad \text{as } a+d \text{ and } (ad-bc) \text{ are just numbers} \\ & \implies 0 = (a+d)A^2 - (ad-bc)A \quad \text{using } A^3 = 0 \\ & \implies A \cdot 0 = (a+d)A^3 - (ad-bc)A^2 \quad \text{multiplying by } A \text{ again} \\ & \implies 0 = (a+d)0 - (ad-bc)A^2 \quad \text{using } A^3 = 0 \\ & \implies (ad-bc)A^2 = 0 \end{aligned}$$

Now, as long as $(ad-bc) \neq 0$, the only way $(ad-bc)A^2 = 0$ can be the zero matrix is if $A^2 = 0$.

iii. $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}^3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = 0$

(b) This is just a matrix multiplication exercise:

$$\begin{aligned} (PQ)^2 &= (PQ)(PQ) \\ &= Q^2PQ^2P \quad \text{using } PQ = Q^2P \\ &= QQPQQP \\ &= QQ(PQ)QP \\ &= QQ(Q^2P)QP \quad \text{again using } PQ = Q^2P \\ &= QQQQPQP \\ &= Q^4(PQ)P \\ &= Q^4(Q^2P)P \quad \text{yet again using } PQ = Q^2P \\ &= Q^4Q^2PP \\ &= Q^6P^2 \end{aligned}$$

6. (**10 Marks**) Recall that the span of a vector space is the set of all linear combinations of its elements. If S, T are subsets of a vector space V, then prove that

(a) $S \subset T$ implies $span(S) \subset span(T)$

(b) $span(span(S)) = span(S)$

Solution:

(i) If $S \subset T$ then every element in S is also in T. Now $span(S)$ contains all the linear combinations of the elements of S. But since T also contains all the elements of S, $span(T)$ also contains all the linear combination of the elements of S. Another way of saying this is: $span(S) \subset span(T)$.

(ii) We need to show (1) $span(span(S)) \subset span(S)$ and (2) $span(S) \subset span(span(S))$

(1) Let $l_1 = \sum_{i=1}^{n} a_i s_i = a_1 s_1 + a_2 s_2 + \cdots + a_n s_n$ and $l_2 = \sum_{i=1}^{n} b_i s_i = b_1 s_1 + b_2 s_2 + \cdots + b_n s_n$ be two linear combinations of the elements of S. Then, the linear combination of l_1 and l_2 is just:

$$c_1 l_1 + c_2 l_2 = \sum_{i=1}^{n} (c_1 a_i + c_2 b_i) s_i = (c_1 a_1 + c_2 b_1) s_1 + (c_1 a_2 + c_2 b_2) s_2 + \cdots + (c_1 a_n + c_2 b_n) s_n$$

which is just a linear combination of the elements of S and so is in $span(S)$. Hence, $span(span(S)) \subset span(S)$.

(2) This is trivial as $span(span(S))$ contains all the elements of $span(S)$. That is, any element, say l_1 of $span(S)$ can be written as a trivial linear combination of the elements in $span(S)$, namely $1 \cdot l_1$.

7. (**10 Marks**) Some Cobb-Douglasing.

(a) The following modified version of the Cobb-Douglas function has been used in some economic studies:

$$F(K, L) = AK^a L^b e^{cK/L} \qquad (A, a, b, \text{ and } c \text{ are positive constants})$$

Compare the marginal products F_K and F_L and discuss their signs.

(b) For the general Cobb-Douglas function F in logarithmic form,

$$\log F = \log A + a_1 \log x_1 + a_2 \log x_2 + \cdots + a_n \log x_n$$

show that

$$\sum_{i=1}^{n} x_i \frac{\partial F}{\partial x_i} = (a_1 + a_2 + \cdots + a_n) F$$

Solution:

(a) Differentiation respect to K we have:

$$F_K = AaK^{a-1} L^b e^{cK/L} + AK^a L^b (c/L) e^{cK/L} > 0$$

because A, a, c, K, L are all positive. So the marginal product of capital is certainly always positive. Similarly, differentiation with respect to L we have:

$$F_L = AbK^a L^{b-1} e^{cK/L} + AK^a L^b (-cK/L^2) e^{cK/L}$$

which is positive if and only if $b > L(cK/L^2) = cK/L$ or $K/L < b/c$. So the marginal product of labour may not always be positive.

(b) Notice that $F = \exp(\log A + a_1 \log x_1 + a_2 \log x_2 + \cdots + a_n \log x_n)$. So,

$$\frac{\partial F}{\partial x_i} = \exp(\log A + a_1 \log x_1 + a_2 \log x_2 + \cdots + a_n \log x_n) \cdot (a_i/x_i)$$

Then

$$\begin{aligned}
\sum_{i=1}^{n} x_i \frac{\partial F}{\partial x_i} &= \sum_{i=1}^{n} x_i \cdot \exp(\log A + a_1 \log x_1 + a_2 \log x_2 + \cdots + a_n \log x_n) \cdot (a_i/x_i) \\
&= \sum_{i=1}^{n} \exp(\log A + a_1 \log x_1 + a_2 \log x_2 + \cdots + a_n \log x_n) \cdot a_i \\
&= \sum_{i=1}^{n} F \cdot a_i \\
&= F(\sum_{i=1}^{n} a_i) \\
&= F(a_1 + a_2 + \cdots + a_n)
\end{aligned}$$

8. (**15 Marks**) Feeling constrained.

(a) Find the values of x, y, z that maximize the function $f(x, y, z) = Ax^a y^b z^c$ subject to $px + qy + rz = m$. (Assume that the constants A, a, b, c, p, q, r, m are all positive and $a + b + c \leq 1$.)

(b) Verify the second-order condition for a maximum.

(c) Using the Envelope Theorem find the following derivatives: $\frac{df^*}{da}, \frac{df^*}{dA}, \frac{df^*}{dq}$ and $\frac{df^*}{dm}$.

Solution:

(a) This a problem with one equality constraint so we form the Lagrangian as follows:

$$\mathfrak{L}(x, y, z, \lambda) = Ax^a y^b z^c - \lambda[px + qy + rz - m]$$

Taking first-order conditions we have:

$$\begin{aligned}
x &: aAx^{a-1}y^b z^c - \lambda p = 0 \\
y &: bAx^a y^{b-1} z^c - \lambda q = 0 \\
z &: cAx^a y^b z^{c-1} - \lambda r = 0 \\
\lambda &: px + qy + rz - m = 0
\end{aligned}$$

Then, solving out for the multiplier in each of the first three equations we have:

$$\lambda = \frac{aAx^{a-1}y^b z^c}{p} = \frac{bAx^a y^{b-1} z^c}{q} = \frac{cAx^a y^b z^{c-1}}{r}$$

Simplifying these three equations we have using the first two:

$$ay/p = bx/q$$

or $y = \frac{bp}{aq}x$. Then using the first and third we have:

$$az/p = cx/r$$

or $z = \frac{cp}{ar}x$. Substituting these two equations in the first-order equation with respect to λ (that is the budget constraint) we have:

$$px + q\frac{bp}{aq}x + r\frac{cp}{ar}x = m \implies x = \frac{m}{p + bp/a + cp/a}$$

Now, solving for the other variables we have:

$$y = \frac{bp}{aq}x = \frac{bpm}{aq(p + bp/a + cp/a)} = \frac{m}{q + aq/b + cq/b}$$
$$z = \frac{cp}{ar}x = \frac{cpm}{ar(p + bp/a + cp/a)} = \frac{m}{r + ar/c + br/c}$$

and notice that the relations for λ this implies that $\lambda > 0$ so that the constraint binds.

(b) (skip - ugly but straight-forward, construct bordered Hessian and then take the plunge!)

(c) Using the Envelope theorem we have:

$$\frac{\partial f^*}{\partial a} = \frac{\partial \mathfrak{L}}{\partial a} = aAx^{a-1}y^b z^c > 0$$
$$\frac{\partial f^*}{\partial A} = \frac{\partial \mathfrak{L}}{\partial A} = x^a y^b z^c > 0$$
$$\frac{\partial f^*}{\partial q} = \frac{\partial \mathfrak{L}}{\partial q} = \lambda x > 0$$
$$\frac{\partial f^*}{\partial m} = \frac{\partial \mathfrak{L}}{\partial m} = \lambda > 0$$

9. (**10 Marks**) Af heals Canada (with apologies to Albertans).
Af gets a call from the Big O(h) who commands Af to start healing Canada. So he gives up his dreams of wearing suit jackets with elbow patches and sipping cherry at post-seminar 'wine and cheese' sessions. Instead, he becomes an environmentalist and moves to Fort McMurray. To heal the planet he needs money so he starts producing electricity from oil sands. He finds that the demand for electricity varies between peak-periods, during which all the generating capacity is used, and off-peak periods. Sales of electric power in n periods are $x_1, x_2, \ldots, x_n$. Trying to comply with local customs, after a careful reading of Mark 12:15, Af finds that Jesus would have wanted the prices in the n periods to be $p_1, p_2, \ldots, p_n$. The total operating cost over all n periods is $C(x_1, \ldots, x_n)$, and k is the output capacity in each period. The cost of maintaining output capacity k is $D(k)$. Af's profit is then

$$\pi(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} p_i x_i - C(x_1, \ldots, x_n) - D(k)$$

Because he cannot exceed capacity k in any period, he faces the constraints

$$x_1 \leq k, x_2 \leq k, \ldots, x_n \leq k$$

Find the sales that will maximize Af's profits and lead to the healing of the planet.

Solution: The Lagrangian is $\mathfrak{L}$ is:

$$\mathfrak{L}(x_1, \dots, x_n, k) = \sum_{i=1}^{n} p_i x_i - C(x_1, \cdots, x_n) - D(k) - \sum_{i=1}^{n} \lambda_i (x_i - k)$$

The choice $(x_1^*, \dots, x_n^*, k^*) \geq 0$ can solve the problem only if there exist Lagrange multipliers $\lambda_1 \geq 0, \dots, \lambda_n \geq 0$ such that

$$\frac{\partial \mathfrak{L}}{\partial x_i} = p_i - C_i'(x_i^*, \dots, x_n^*) - \lambda_i \leq 0 \ (= 0 \text{ if } x_i^* > 0) \tag{1}$$

$$\frac{\partial \mathfrak{L}}{\partial k} = -D'(k^*) + \sum_{i=1}^{n} \lambda_i \leq 0 \ (= 0 \text{ if } k^* > 0) \tag{2}$$

$$\lambda_i \geq 0 \ (= 0 \text{ if } x_i^* < k^*) \tag{3}$$

Suppose i is such that $x_i^* > 0$. Then implies that:

$$p_i = C_i'(x_1^*, \dots, x_n^*) + \lambda_i \tag{4}$$

If period i is an off-peak period, then $x_i^* < k^*$ and so $\lambda_i = 0$ by (3). From (4), it follows that $p_i = C_0'(x_1^*, \dots, x_n^*)$. Thus, we see that the *profit maximizing pattern of outputs* $(x_1^*, \dots, x_n^*)$ will bring about equality between the price in any off-peak period and the corresponding marginal operating cost. On the other hand, λ_j might be positive in a peak period when $x_j^* = k^*$. If $k^* > 0$, if follows from (2) that $\sum_{i=1}^{n} \lambda_i = D'(k^*)$.

We conclude that the output pattern will be such that in peak periods the price set will exceed the marginal operating cost by an additional amount λ_i, which is really the "shadow price" of the capacity constraint $x_i \leq k$. The sum of these shadow prices over all peak periods is equal to the marginal capacity cost.