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question paper

Student Number: ______________
Student Name:________________

Wilfrid Laurier UNIVERSITY

School of Business and Economics

EC 450 – Advanced Macroeconomics

Suggested Solutions to Midterm Examination

March 13, 2008

Instructor: Sharif F. Khan

Time Limit: 1 Hour 20 Minutes

Instructions:

Important! Read the instructions carefully before you start your exam.

Write your answers in the answer booklets provided. Record your full name and student number on both the question and answer booklets.

Marking Scheme:

Part A [20 marks] Two of Three True/ False/Uncertain questions – 10 marks each

Part B [30 marks] One problem solving / analytical question questions

Calculators:

Non-programmable calculators are permitted

Part A **[20 marks]**

Answer two of the following three questions. Each question is worth 10 marks.

A1. Consider a general Solow economy. Assume that the production function is Cobb-Douglas:

$$Y_t = B_t K_t^{\alpha} L_t^{1-\alpha} \qquad 0 < \alpha < 1,$$

where *Y* is aggregate output, *K* is the stock of aggregate capital, *L* is total labor and *B* is the total factor productivity. Assume that *L* and *B* grow exogenously at constant rates *n* and *g*, respectively. Suppose the growth accounting techniques are applied to this economy.

(a) On the balanced growth path of this Solow economy, what fraction of growth in output per worker does growth accounting attribute to growth in capital per worker? What fraction does it attribute to technological progress? [5 marks]

Define $y_t \equiv \frac{Y_t}{L_t}$ and $k_t \equiv \frac{K_t}{L_t}$. Thus, y_t denotes denote output per worker and k_t denote capital per worker.

Production function: $Y_t = B_t K_t^{\alpha} L_t^{1-\alpha}$ (1)

Dividing both sides of the production function, equation (1), by L_t we get:

$$\frac{Y_t}{L_t} = \frac{B_t K_t^{\alpha} L_t^{1-\alpha}}{L_t}$$

$$\Rightarrow \frac{Y_t}{L_t} = \frac{B_t K_t^{\alpha}}{L_t^{1-1+\alpha}}$$

$$\Rightarrow \frac{Y_t}{L_t} = \frac{B_t K_t^{\alpha}}{L_t^{\alpha}}$$

$$\Rightarrow \frac{Y_t}{L_t} = B_t \left(\frac{K_t}{L_t}\right)^{\alpha}$$

$$\Rightarrow y_t = B_t k_t^{\alpha} \qquad (2)$$

[Using the definition of y_t and k_t]

For another and later year $T > t$, we have:

$$y_T = B_T k_T^{\alpha} \qquad (3)$$

If we take logs on both sides of (2) and (3), we get:

$$\ln y_t = \ln B_t + \alpha \ln k_t \qquad (4)$$

$$\ln y_T = \ln B_T + \alpha \ln k_T \qquad (5)$$

Subtracting (4) from (5), and divide on both sides by *T-t*, we get:

$$\frac{\ln y_T - \ln y_t}{T-t} = \frac{\ln B_T - \ln B_t}{T-t} + \alpha \frac{\ln k_T - \ln kt}{T-t} \qquad (6)$$

Let g^y denote the growth rate in output per worker, g^B denote the growth rate of B_t and g^k denote the growth rate in capital per worker. So, we can rewrite (6) as,

$$g^y = g^B + \alpha g^k \qquad (7)$$

The growth-accounting equation (7) splits up the growth in GDP per worker into contributions from growth in capital per worker and growth in the total factor productivity (technological progress). Now imagine applying this growth-accounting equation to a Solow economy that is on its balanced growth path. On the balanced growth path, the growth rates of output per worker and capital per worker are both equal to *g*, the growth rate of *B*. Thus equation (1) implies that growth accounting would attribute a fraction α of growth in output per worker to growth in capital per worker. It would attribute the rest – fraction $(1-\alpha)$ - to technological progress. So, with our usual estimate of $\alpha = 1/3$, growth accounting would attribute 67 percent of the growth in output per worker to technological progress and about 33 percent of the growth in output per worker to growth in capital per worker.

(b) How can you reconcile your results in (a) with the fact that the general Solow model implies that the growth rate of output per worker on the balanced growth path is determined solely by the rate of technological progress? [5 marks]

In an accounting sense, the result in part **(a)** would be true, but in deeper sense it would not: the reason that the capital-labor ratio grows at rate *g* on the balanced growth path is because the total factor productivity is growing at rate *g*. That is, the growth in the total factor productivity – the growth in *B* – raises output per worker through two channels. It raises output per worker not only by directly raising output but also by (for a given saving rate) increasing the resources devoted to capital accumulation and thereby raising the capital-labor ratio. Growth accounting attributes the rise in output per worker through the second channel to growth in the capital-labor ratio, not to its underlying source. Thus, although growth accounting is often instructive, it is not appropriate to interpret it as shedding light on the underlying determinants of growth.

A2. Consider the Solow model with human capital. Assume that the economy has a Cobb-Douglas aggregate production function with labor-augmenting technological progress:

$$Y_t = K_t^{\alpha} H_t^{\varphi} \left(A_t L_t\right)^{1-\alpha-\varphi}, \qquad 0<\alpha<1, \qquad 0<\varphi<1, \qquad \alpha+\varphi<1,$$

where Y is aggregate output, K is the stock of aggregate physical capital, H the stock of aggregate capital, L is total labor and A is the effectiveness of labor. Assume that L and A grow exogenously at constant rates n and g, respectively. Capital depreciates at a constant rate δ.

In this economy, hiring one more (marginal) unit of labor means hiring one more unit endowed with the average amount, h_t, of human capital per worker. Hence, a firm cannot increase the input of 'raw' labor, L_t, without increasing proportionally the input of human capital, $H_t = h_t L_t$.

Denote with lower case letters the variables in unit of effective worker. That means,

$$\tilde{y} \equiv \frac{Y}{AL}, \ \tilde{k} \equiv \frac{K}{AL} \ \text{and} \ \tilde{h} \equiv \frac{H}{AL}.$$

It can be shown that the physical capital per effective worker and human capital per effective worker in this economy evolve according to the following two Solow equations, respectively:

$$\tilde{k}_{t+1} - \tilde{k}_t = \frac{1}{(1+n)(1+g)}\left(s_K \tilde{k}_t^{\alpha} \tilde{h}_t^{\varphi} - (n+g+\delta+ng)\tilde{k}_t\right) \ \text{and}$$

$$\tilde{h}_{t+1} - \tilde{h}_t = \frac{1}{(1+n)(1+g)}\left(s_H \tilde{k}_t^{\alpha} \tilde{h}_t^{\varphi} - (n+g+\delta+ng)\tilde{h}_t\right).$$

It can be also shown that the steady-state equilibrium values of physical capital per effective worker and human capital per effective worker in this economy are:

$$\tilde{k}^* = \left(\frac{s_K^{1-\varphi} s_H^{\varphi}}{n+g+\delta+ng}\right)^{\frac{1}{1-\alpha-\varphi}}, \ \tilde{h}^* = \left(\frac{s_K^{\alpha} s_H^{1-\alpha}}{n+g+\delta+ng}\right)^{\frac{1}{1-\alpha-\varphi}} \ \text{and}$$

$$\tilde{y}^* = \left(\frac{s_K}{n+g+\delta+ng}\right)^{\frac{\alpha}{(1-\alpha-\varphi)}} \left(\frac{s_H}{n+g+\delta+ng}\right)^{\frac{\varphi}{(1-\alpha-\varphi)}}.$$

(a) Derive the equation which gives all combinations of $\tilde{k}_t$, $\tilde{h}_t$ such that $\tilde{k}_t$ stays unchanged. Label this equation as "$[\Delta\tilde{k}_t = 0]$" equation. Derive the equation which gives all combinations of $\tilde{k}_t$, $\tilde{h}_t$ such that $\tilde{h}_t$ stays unchanged. Label this equation as "$[\Delta\tilde{h}_t = 0]$" equation. Draw a phase diagram by plotting $[\Delta\tilde{k}_t = 0]$ equation and $[\Delta\tilde{h}_t = 0]$ equation. Clearly the identify the dynamics of $\tilde{k}_t$ and $\tilde{h}_t$ in all regions of the diagram. Provide explanation for your answer. [5 marks]

It is given that the physical capital per effective worker and human capital per effective worker in this economy evolve according to the following two Solow equations, respectively:

$$\tilde{k}_{t+1} - \tilde{k}_t = \frac{1}{(1+n)(1+g)}\left(s_K \tilde{k}_t^{\alpha} \tilde{h}_t^{\varphi} - (n + g + \delta + ng)\tilde{k}_t\right) \text{ and} \quad (1)$$

$$\tilde{h}_{t+1} - \tilde{h}_t = \frac{1}{(1+n)(1+g)}\left(s_H \tilde{k}_t^{\alpha} \tilde{h}_t^{\varphi} - (n + g + \delta + ng)\tilde{h}_t\right). \quad (2)$$

To derive the $[\Delta\tilde{k}_t = 0]$ equation we set $\tilde{k}_{t+1} = \tilde{k}_t$ in equation (1):

$$0 = s_K \tilde{k}_t^{\alpha} \tilde{h}_t^{\varphi} - (n + g + \delta + ng)\tilde{k}_t$$
$$\Rightarrow s_K \tilde{k}_t^{\alpha} \tilde{h}_t^{\varphi} = (n + g + \delta + ng)\tilde{k}_t$$
$$\Rightarrow \tilde{h}_t^{\varphi} = \left[\frac{n + g + \delta + ng}{s_K}\right]\frac{\tilde{k}_t}{\tilde{k}_t^{\alpha}}$$
$$\Rightarrow \tilde{h}_t^{\varphi} = \left[\frac{n + g + \delta + ng}{s_K}\right]\tilde{k}_t^{(1-\alpha)}$$
$$\Rightarrow \tilde{h}_t = \left[\frac{n + g + \delta + ng}{s_K}\right]^{\frac{1}{\phi}} \tilde{k}_t^{\frac{(1-\alpha)}{\phi}} \quad (3)$$

Substituting empirically reasonable values of α and ϕ, $\alpha = \phi = \frac{1}{3}$, into (3) we get:

$$\tilde{h}_t = \left[\frac{n + g + \delta + ng}{s_K}\right]^{3} \tilde{k}_t^{2} \quad (4)$$

If we plot equation (4) on a diagram, we will get a convex-shaped curve $[\Delta\tilde{k}_t = 0]$. This curve gives all the combinations of $\tilde{k}_t$ and $\tilde{h}_t$ such that $\tilde{k}_t$ stays unchanged. We draw this curve in Figure A2(a).

Equation (1) implies that $\tilde{k}_{t+1} - \tilde{k}_t > 0$ if

$$
\begin{aligned}
& s_K \tilde{k}_t^{\alpha} \tilde{h}_t^{\varphi} - (n + g + \delta + ng)\tilde{k}_t > 0 \\
& \Rightarrow s_K \tilde{k}_t^{\alpha} \tilde{h}_t^{\varphi} > (n + g + \delta + ng)\tilde{k}_t \\
& \Rightarrow \tilde{h}_t > \left[\frac{n + g + \delta + ng}{s_K}\right]^{\frac{1}{\phi}} \tilde{k}_t^{\frac{(1-\alpha)}{\phi}} .
\end{aligned}
$$

This means that for every combination of $\tilde{k}_t$ and $\tilde{h}_t$ above the curve $[\Delta\tilde{k}_t = 0]$, $\tilde{k}_t$ must be increasing, while below the same curve it must be decreasing. Increase in $\tilde{k}_t$ is indicated by rightward arrows and decrease in $\tilde{k}_t$ is indicated by leftward arrows in Figure A2(a).

To derive the $[\Delta\tilde{h}_t = 0]$ equation we set $\tilde{h}_{t+1} = \tilde{h}_t$ in equation (2):

$$
\begin{aligned}
& 0 = s_H \tilde{k}_t^{\alpha} \tilde{h}_t^{\varphi} - (n + g + \delta + ng)\tilde{h}_t \\
& \Rightarrow (n + g + \delta + ng)\tilde{h}_t = s_H \tilde{k}_t^{\alpha} \tilde{h}_t^{\varphi} \\
& \Rightarrow \frac{\tilde{h}_t}{\tilde{h}_t^{\varphi}} = \left[\frac{s_H}{n + g + \delta + ng}\right] \tilde{k}_t^{\alpha} \\
& \Rightarrow \tilde{h}_t^{1-\varphi} = \left[\frac{s_H}{n + g + \delta + ng}\right] \tilde{k}_t^{\alpha} \\
& \Rightarrow \tilde{h}_t = \left[\frac{s_H}{n + g + \delta + ng}\right]^{\frac{1}{1-\phi}} \tilde{k}_t^{\frac{\alpha}{1-\phi}} \qquad (5)
\end{aligned}
$$

Substituting empirically reasonable values of α and ϕ, $\alpha = \phi = \frac{1}{3}$, into (5) we get:

$$
\tilde{h}_t = \left[\frac{s_H}{n + g + \delta + ng}\right]^{\frac{3}{2}} \tilde{k}_t^{\frac{1}{2}} \qquad (6)
$$

Now, if we plot equation (6) on a diagram, we will get a concave-shaped curve $[\Delta\tilde{h}_t = 0]$. This curve gives all the combinations of $\tilde{k}_t$ and $\tilde{h}_t$ such that $\tilde{h}_t$ stays unchanged. We draw this curve in Figure A2(a).

Equation (2) implies that $\tilde{h}_{t+1} - \tilde{h}_t > 0$ if

$$
\begin{aligned}
& s_H \tilde{k}_t^{\alpha} \tilde{h}_t^{\varphi} - (n + g + \delta + ng)\tilde{h}_t > 0 \\
& \Rightarrow (n + g + \delta + ng)\tilde{h}_t < s_H \tilde{k}_t^{\alpha} \tilde{h}_t^{\varphi} \\
& \Rightarrow \tilde{h}_t < \left[\frac{s_H}{n + g + \delta + ng}\right]^{\frac{1}{1-\phi}} \tilde{k}_t^{\frac{\alpha}{1-\phi}}.
\end{aligned}
$$

This means that for every combination of $\tilde{k}_t$ and $\tilde{h}_t$ below the curve $[\Delta\tilde{h}_t = 0]$, $\tilde{h}_t$ must be increasing, while above the same curve it must be decreasing. Increase in $\tilde{h}_t$ is indicated by upward arrows and decrease in $\tilde{h}_t$ is indicated by downward arrows in Figure A2(a) which is called a phase diagram.

(b) Illustrate the steady-state equilibrium values of physical capital per effective worker and human capital per effective worker on the phase diagram you drew in part (a). [1 mark]

In Figure A2(a), the intersection point *E* between the two curves $[\Delta\tilde{k}_t = 0]$ and $[\Delta\tilde{h}_t = 0]$ illustrates the steady-state equilibrium values of physical capital per effective worker, $\tilde{k}^*$, and human capital per effective worker, $\tilde{h}^*$.

(c) Using the phase diagram explain the effects of an increase in the physical capital investment rate, s_K, on the steady-state equilibrium values of physical capital per effective worker and human capital per effective worker. Briefly explain how the economy will converge to new steady-state equilibrium after an increase in s_K. What will be the effects on the growth rate of output per worker, y_t, in the short-run and in the long-run? [4 marks]

Assume that the economy is initially in steady state as illustrated by the intersection point E between the two curves $[\Delta\tilde{k}_t = 0]$ and $[\Delta\tilde{h}_t = 0]$ in Figure A2(c). An increase in the physical capital investment rate, s_K, other parameters remaining constant, will decrease the value of the constant term, $\left(\dfrac{n+g+\delta+ng}{s_K}\right)$, on the right side of equation (4). As a result, the curve $[\Delta\tilde{k}_t = 0]$ will shift downward to $[\Delta\tilde{k}_t = 0]'$, as illustrated in Figure A2(c). As the phase diagram has now changed (arrows in Figure A2(c) are drawn considering the new curve $[\Delta\tilde{k}_t = 0]'$), the old steady state point E will be situated exactly on a boundary between two regions, and on this boundary $\tilde{h}_t$ does not change while $\tilde{k}_t$ increases. Once $\tilde{k}_t$ has increased, the economy will be in the region where both $\tilde{k}_t$ and $\tilde{h}_t$ increase and there could be convergence to the new steady state E' along the trajectory indicated in Figure A2(c). Thus, the steady-state equilibrium value of physical capital per effective worker will increase from $\tilde{k}^*$ to $\tilde{k}^{*'}$. The steady-state equilibrium value of human capital per effective worker will increase from $\tilde{h}^*$ to $\tilde{h}^{*'}$.

What happens is that the increased s_K in the very beginning implies more accumulation of physical capital (and only that), but as soon as the increased stock of physical capital begins to generate increases in output, more human capital will also be accumulated because of the constant rate of investment in human capital. This explains why, along the indicated trajectory, both $\tilde{k}_t$ and $\tilde{h}_t$ increase.

The long-run growth rate of output per worker, y_t, will remain unchanged at g, the growth rate of the labor productivity variable A_t. But in short-run, while the economy is in transition from the initial steady state equilibrium E to new steady state equilibrium E', y_t will grow at a higher rate than g.

For question A3, explain why the following statement is True, False, or Uncertain according to economic principles. Use diagrams and/or numerical examples where appropriate. Unsupported answers will receive no marks. It is the explanation that is important.

A3. Output per worker grows at a positive and constant rate at the steady-state equilibrium of Solow growth model.

Uncertain

In the basic Solow model with no technological progress, there is no growth in output per worker at the steady-state equilibrium. However, in the general Solow model with a positive exogenous technological progress, there is a positive and constant growth in output per worker at the steady-state equilibrium.

Figure A3(a) shows the steady-state equilibrium of the basic Solow model where the economy has a Cobb-Douglas production function. The dynamics of the basic Solow model are such that from any strictly positive initial value, k_0, the level of capital per worker, $k_t \equiv \frac{K_t}{L_t}$, will converge monotonically to its steady state value, k^*, as illustrated by point *E* in Figure A3(a). At the steady-state equilibrium *E*, the actual savings, $sB(k^*)^\alpha$, equals the break-even investment, $(n+\delta)k^*$, which is required to keep the level of capital per worker unchanged at k^*. There is an associated steady state value for output per worker, $y^* = B(k^*)^\alpha$, and $y_t \equiv \frac{Y_t}{L_t}$ converges to y^* over time. Once the economy converges to the steady-state equilibrium, the level of capital per worker remains unchanged at its steady state value, $k^* = B^{\frac{1}{1-\alpha}}\left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}}$ and the output per worker remains constant at its steady state value, $y^* = B^{\frac{1}{1-\alpha}}\left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$. It implies that at the steady-state equilibrium, the growth rates of capital per worker and output per worker are both zero.

Figure A3(b) shows the steady-state equilibrium of the general Solow model where the economy has a Cobb-Douglas production function with labor-augmenting technological progress. The dynamics of the general Solow model are such that from any strictly positive initial value, $\tilde{k}_0$, the level of capital per effective worker, $\tilde{k}_t = \frac{k_t}{A_t}$, will converge monotonically to its steady state value, $\tilde{k}^*$, as illustrated by point *E* in Figure A3(b). At the steady-state equilibrium *E*, the actual savings, $sB(\tilde{k}^*)^\alpha$, equals the break-even investment, $(n+g+\delta+ng)\tilde{k}^*$, which is required to keep the level of capital per effective worker unchanged at $\tilde{k}^*$. There is an associated steady state value for output per effective worker, $\tilde{y}^* = B\left(\tilde{k}^*\right)^\alpha$, and $\tilde{y}_t \equiv \frac{y_t}{A_t}$ converges to $\tilde{y}^*$ over time. Once the economy converges to the steady-state equilibrium, the level of capital per effective worker remains unchanged at its steady state value, $\tilde{k}^* = \left(\frac{s}{n+g+\delta+ng}\right)^{\frac{1}{1-\alpha}}$ and the output-per effective worker remains constant at its steady state value, $\tilde{y}^* = \left(\frac{s}{n+g+\delta+ng}\right)^{\frac{\alpha}{1-\alpha}}$. This means that at the steady-state equilibrium, the growth rates of capital per effective worker and output per effective worker are both zero. This implies that at the steady-state equilibrium both capital per worker, k_t, and output per worker, y_t, grows at the same rate, namely the growth rate g of the labor productivity variable, A_t, since otherwise capital per effective worker, $\frac{k_t}{A_t}$, and output per effective worker, $\frac{y_t}{A_t}$, could not be constant.

Figure A3(c) shows the steady-state equilibrium growth path of y_t where output per worker grows at a constant and positive rate of g.

Part B Problem Solving Questions [30 marks]

Read each part of the question very carefully. Show all the steps of your calculations to get full marks.

B1. [30 Marks]

Consider a general Solow economy with labor-augmenting technological progress.

$Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}$, $\qquad 0 < \alpha < 1$,

where *Y* is aggregate output, *K* is the stock of aggregate capital, *L* is total labor and *A* is the effectiveness of labor. Assume that *L* and *A* grow exogenously at constant rates *n* and *g*, respectively. Capital depreciates at a constant rate δ. Denote with lower case letters the variables in unit of effective labor. That means,

$\tilde{y} \equiv \frac{Y}{AL}$ **and** $\tilde{k} \equiv \frac{K}{AL}$.

The evolution of aggregate capital in the economy is given by

$$K_{t+1} - K_t = sY_t - \delta K_t$$

where s is a constant and exogenous saving rate.

(a) Derive the law of motion, or the transition equation, for capital per effective worker. What is the economic interpretation of this equation? Plot the transition equation in a diagram. Clearly identify the steady-state equilibrium level of capital per effective worker in this diagram. [10 marks] [Note: You do NOT have to explain the process of convergence to steady state]

Define $\tilde{y}_t \equiv \frac{Y_t}{A_t L_t}$ and $\tilde{k}_t \equiv \frac{K_t}{A_t L_t}$. Thus, $\tilde{y}_t$ denotes denote output per effective worker and $\tilde{k}_t$ denote capital per effective worker.

Production function: $Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}$ (1)

An exogenous growth in labor: $L_{t+1} = (1+n)L_t$ (2)

An exogenous growth in the effectiveness of labor: $A_{t+1} = (1+g)A_t$ (3)

The evolution equation of aggregate capital: $K_{t+1} - K_t = sY_t - \delta K_t$ (4)

Dividing both sides of the production function, equation (1), by $A_t L_t$ we get:

$$\frac{Y_t}{A_t L_t} = \frac{K_t^{\alpha}(A_t L_t)^{1-\alpha}}{A_t L_t}$$

$$\Rightarrow \frac{Y_t}{A_t L_t} = \frac{K_t^{\alpha}}{(A_t L_t)^{1-1+\alpha}}$$

$$\Rightarrow \frac{Y_t}{A_t L_t} = \frac{K_t^{\alpha}}{(A_t L_t)^{\alpha}}$$

$$\Rightarrow \frac{Y_t}{A_t L_t} = \left(\frac{K_t}{A_t L_t}\right)^{\alpha}$$

$$\Rightarrow \tilde{y}_t = \tilde{k}_t^{\alpha} \qquad (5) \qquad \text{[Using the definitions of } \tilde{y}_t \text{ and } \tilde{k}_t\text{]}$$

Rewrite equation (4) as:

$$K_{t+1} = sY_t + (1-\delta)K_t . \qquad (4.1)$$

Dividing both sides of equation (4.1) by $A_{t+1}L_{t+1}$ we get:

$$\frac{K_{t+1}}{A_{t+1}L_{t+1}} = \frac{sY_t + (1-\delta)K_t}{A_{t+1}L_{t+1}}$$

$$\Rightarrow \frac{K_{t+1}}{A_{t+1}L_{t+1}} = \frac{sY_t + (1-\delta)K_t}{(1+g)A_t(1+n)L_t} \qquad \text{[Using equations (2) and (3)]}$$

$$\Rightarrow \frac{K_{t+1}}{A_{t+1}L_{t+1}} = \frac{1}{(1+n)(1+g)}\left(s\frac{Y_t}{A_t L_t} + (1-\delta)\frac{K_t}{A_t L_t}\right)$$

$$\Rightarrow \tilde{k}_{t+1} = \frac{1}{(1+n)(1+g)}\left(s\tilde{y}_t + (1-\delta)\tilde{k}_t\right) \qquad \text{[Using the definitions of } \tilde{y}_t \text{ and } \tilde{k}_t\text{]}$$

$$\Rightarrow \tilde{k}_{t+1} = \frac{1}{(1+n)(1+g)}\left(s\tilde{k}_t^{\alpha} + (1-\delta)\tilde{k}_t\right) \qquad \text{[Substituting } \tilde{y}_t \text{ using (5)]}$$

Therefore, the transition equation for capital per effective worker is:

$$\tilde{k}_{t+1} = \frac{1}{(1+n)(1+g)}\left(s\tilde{k}_t^{\alpha} + (1-\delta)\tilde{k}_t\right) \qquad (6)$$

The transition equation tells us how capital effective worker, $\tilde{k}_t$, evolves over time from a given initial positive value of $\tilde{k}_t$, $\tilde{k}_0$. For a given initial positive value, $\tilde{k}_0$, of capital per effective worker in year zero, (6) determines capital per effective worker, $\tilde{k}_1$, of year one, which then can be inserted on the right-hand side to determine $\tilde{k}_2$, etc. In this way, given $\tilde{k}_0$, the transition equation determines the full dynamic sequence of capital per effective worker.

Figure B1(a) shows the transition equation as given by (6). This curve starts at (0,0) and is everywhere increasing. The 45^0 line, $k_{t+1} = k_t$, has also been drawn.

Differentiating (6) gives:

$$\frac{d\tilde{k}_{t+1}}{d\tilde{k}_t} = \frac{s\lambda A\tilde{k}_t^{\alpha-1} + (1-\delta)}{(1+n)(1+g)}.$$

This shows that the slope of the transition curve decreases monotonically from infinity to $(1-\delta)/(1+n)(1+g)$, as $\tilde{k}_t$ increases from zero to infinity. The latter slope is positive and less than one if $n + g + \delta + ng > 0$, which is empirically plausible. Hence the transition curve must have a unique intersection with the 45^0 line (which has slope one), to the right of $\tilde{k}_t = 0$.

The intersection between the transition curve and the 45^0 line is the unique positive solution, $\tilde{k}^*$, which is obtained by setting $\tilde{k}_{t+1} = \tilde{k}_t = \tilde{k}$ in (6) and solving for $\tilde{k}$. This $\tilde{k}^*$ is called the steady state value of capital per effective worker. The level of $\tilde{k}^*$ is clearly identified in Figure B1(a).

(b) Solve for the steady state equilibrium values of capital per effective worker, output per effective worker, consumption per effective worker. Illustrate the steady-state equilibrium values of capital per effective worker, output per effective worker and consumption per effective worker in a diagram. [10 marks]

To solve for the steady state equilibrium value of aggregate capital per effective worker we set $\tilde{k}_{t+1} = \tilde{k}_t = \tilde{k}$ in (6),

$$\tilde{k}_{t+1} = \frac{1}{(1+n)(1+g)}\left(s\tilde{k}_t^{\alpha} + (1-\delta)\tilde{k}_t\right)$$

$$\Rightarrow \tilde{k} = \frac{1}{(1+n)(1+g)}\left(s\tilde{k}^{\alpha} + (1-\delta)\tilde{k}\right)$$

$$\Rightarrow \tilde{k}(1+n)(1+g) = s\tilde{k}^{\alpha} + (1-\delta)\tilde{k}$$

$$\Rightarrow \tilde{k} + n\tilde{k} + g\tilde{k} + ng\tilde{k} = s\tilde{k}^{\alpha} + \tilde{k} - \delta\tilde{k}$$

$$\Rightarrow \tilde{k} + n\tilde{k} + g\tilde{k} + ng\tilde{k} - \tilde{k} + \delta\tilde{k} = s\tilde{k}^{\alpha}$$

$$\Rightarrow (n+g+\delta+ng)\tilde{k} = s\tilde{k}^{\alpha}$$

$$\Rightarrow \frac{\tilde{k}}{\tilde{k}^{\alpha}} = \frac{s}{(n+g+\delta+ng)}$$

$$\Rightarrow \tilde{k}^{(1-\alpha)} = \left(\frac{s}{n+g+\delta+ng}\right)$$

$$\Rightarrow \left(\tilde{k}^{(1-\alpha)}\right)^{\frac{1}{1-\alpha}} = \left(\frac{s}{n+g+\delta+ng}\right)^{\frac{1}{1-\alpha}}$$

$$\therefore \tilde{k}^{*} = \left(\frac{s}{n+g+\delta+ng}\right)^{\frac{1}{1-\alpha}} \qquad (7)$$

Therefore, the steady state equilibrium value of aggregate capital per effective worker is $\left(\frac{s}{n+g+\delta+ng}\right)^{\frac{1}{1-\alpha}}$.

The steady state equilibrium value of aggregate output per effective worker, $\tilde{y}^{*}$, is:

$$\tilde{y}^{*} = \tilde{k}^{*\alpha}$$

$$\Rightarrow \tilde{y}^{*} = \left[\left(\frac{s}{n+g+\delta+ng}\right)^{\frac{1}{1-\alpha}}\right]^{\alpha}$$

$$\therefore \tilde{y}^{*} = \left(\frac{s}{n+g+\delta+ng}\right)^{\frac{\alpha}{1-\alpha}}. \qquad (8)$$

Consumption per effective worker is $\tilde{c}_t = (1-s)\tilde{y}_t$ in any period t. So, the steady state equilibrium value of consumption per effective worker is:

$$\tilde{c}^{*} = (1-s)\tilde{y}^{*} = (1-s)\left(\frac{s}{n+g+\delta+ng}\right)^{\frac{\alpha}{1-\alpha}}. \qquad (9)$$

Figure B1(b) illustrates the steady state equilibrium levels of capital per effective worker, output per effective worker and consumption per effective worker.

(c) Find the growth rate of output per worker at the steady state equilibrium. [2 marks]

At the steady state equilibrium the level of aggregate output per effective worker is: $\tilde{y}^* = \left(\frac{s}{n+g+\delta+ng}\right)^{\frac{\alpha}{1-\alpha}}$, which is a constant. So, the growth rate of $\tilde{y}^*$ is zero.

Define $y_t \equiv \frac{Y_t}{N_t}$ and $k_t \equiv \frac{K_t}{N_t}$. Thus, y_t denotes denote output per worker and k_t denote capital per worker. Let the approximate growth rates of A_t, $\tilde{y}_t$, y_t from period $t-1$ to t be denoted by g_t^A, $g_t^{\tilde{y}}$ and g_t^y, respectively. Using the definitions of $\tilde{y}_t$ and y_t we can express output per worker as,

$$y_t = \tilde{y}_t A_t \qquad (10)$$

Taking logs on both sides of (10),

$$\ln y_t = \ln \tilde{y}_t + \ln A_t \qquad (11)$$

So, we can also write,

$$\ln y_{t-1} = \ln \tilde{y}_{t-1} + \ln A_{t-1} \qquad (12)$$

Subtracting (12) from (11) we get,

$$\ln y_t - \ln y_{t-1} = \ln \tilde{y}_t - \ln \tilde{y}_{t-1} + \ln A_t - \ln A_{t-1}$$

$$\Rightarrow g_t^y = g_t^{\tilde{y}} + g_t^A \qquad (13)$$

$$\Rightarrow g_t^y = g_t^{\tilde{y}} + g$$

Since at the steady state equilibrium $g_t^{\tilde{y}}$ is zero, the growth rate of output per worker, g_t^y, at the steady state equilibrium must be equal to growth in the effectiveness of labor, g.

[Note: Using the rule of thumb, we can also get equation (13) directly from equation (10). So, it is NOT necessary to show the intermediate steps between equation (10) and equation (13).]

(d) Find the golden rule savings rate. Find the golden rule levels of capital per effective worker and consumption effective per worker. Illustrate the golden rule levels of capital per effective worker and consumption per effective worker in a diagram. [8 marks]

The steady state equilibrium value of consumption per effective worker is:

$$\tilde{c}^* = (1-s)\left(\frac{s}{n+g+\delta+ng}\right)^{\frac{\alpha}{1-\alpha}}. \tag{9}$$

The golden rule saving rate, s^{**}, is the steady-state consumption per effective worker maximizing saving rate. So, to find s^{**} we have to maximize $\tilde{c}^*$ with respect to s using (9). We are allowed to take logs before maximizing. Taking logs on both sides of (9) gives:

$$\ln\tilde{c}^* = \ln(1-s) + \frac{\alpha}{1-\alpha}\ln s - \frac{\alpha}{1-\alpha}\ln(n+g+\delta+ng).$$

So, the first-order condition of this maximization problem:

$$\frac{\partial \ln\tilde{c}^*}{\partial s} = -\frac{1}{1-s} + \left(\frac{\alpha}{1-\alpha}\right)\frac{1}{s} = 0$$
$$\Rightarrow \left(\frac{\alpha}{1-\alpha}\right)\frac{1}{s} = \frac{1}{1-s}$$
$$\Rightarrow \alpha(1-s) = s(1-\alpha)$$
$$\Rightarrow \alpha - \alpha s = s - \alpha s$$
$$\therefore\ s^{**} = \alpha.$$

So, the golden rule saving rate is α.

To find the golden rule level of capital per effective worker, $\tilde{k}^{**}$, we have to substitute s with α in (7):

$$\tilde{k}^{**} = \left(\frac{\alpha}{n+g+\delta+ng}\right)^{\frac{1}{1-\alpha}}$$

To find the golden rule level of consumption per effective worker, $\tilde{c}^{**}$, we have to substitute s with α in (9):

$$\tilde{c}^{**} = (1-\alpha)\left(\frac{\alpha}{n+g+\delta+ng}\right)^{\frac{\alpha}{1-\alpha}}.$$

Figure B1(d) illustrates the golden rule levels of capital per effective worker and consumption per effective worker.

FIGURE A2(a): Phase Diagram for the Solow model with human capital

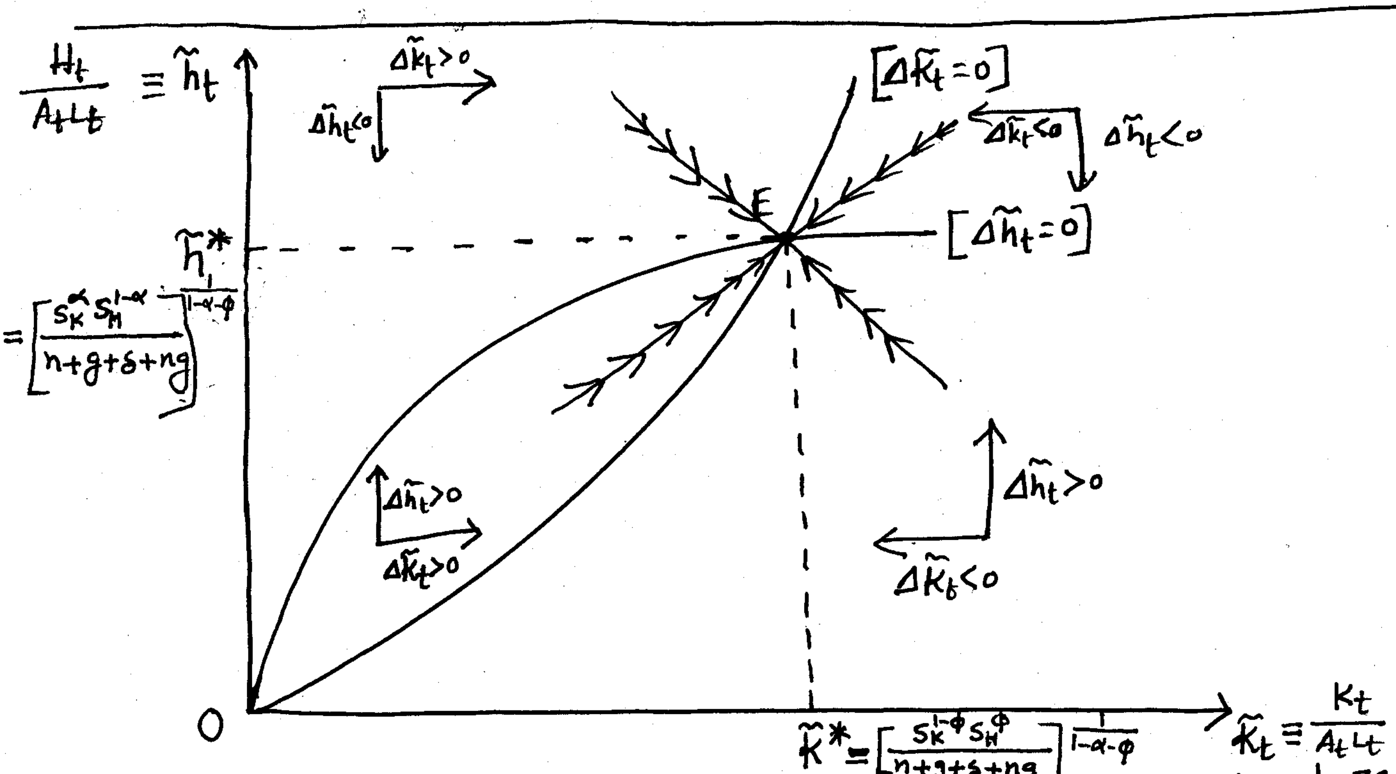

FIGURE A2(c): Effects of an increase in the physical investment rate, s_K

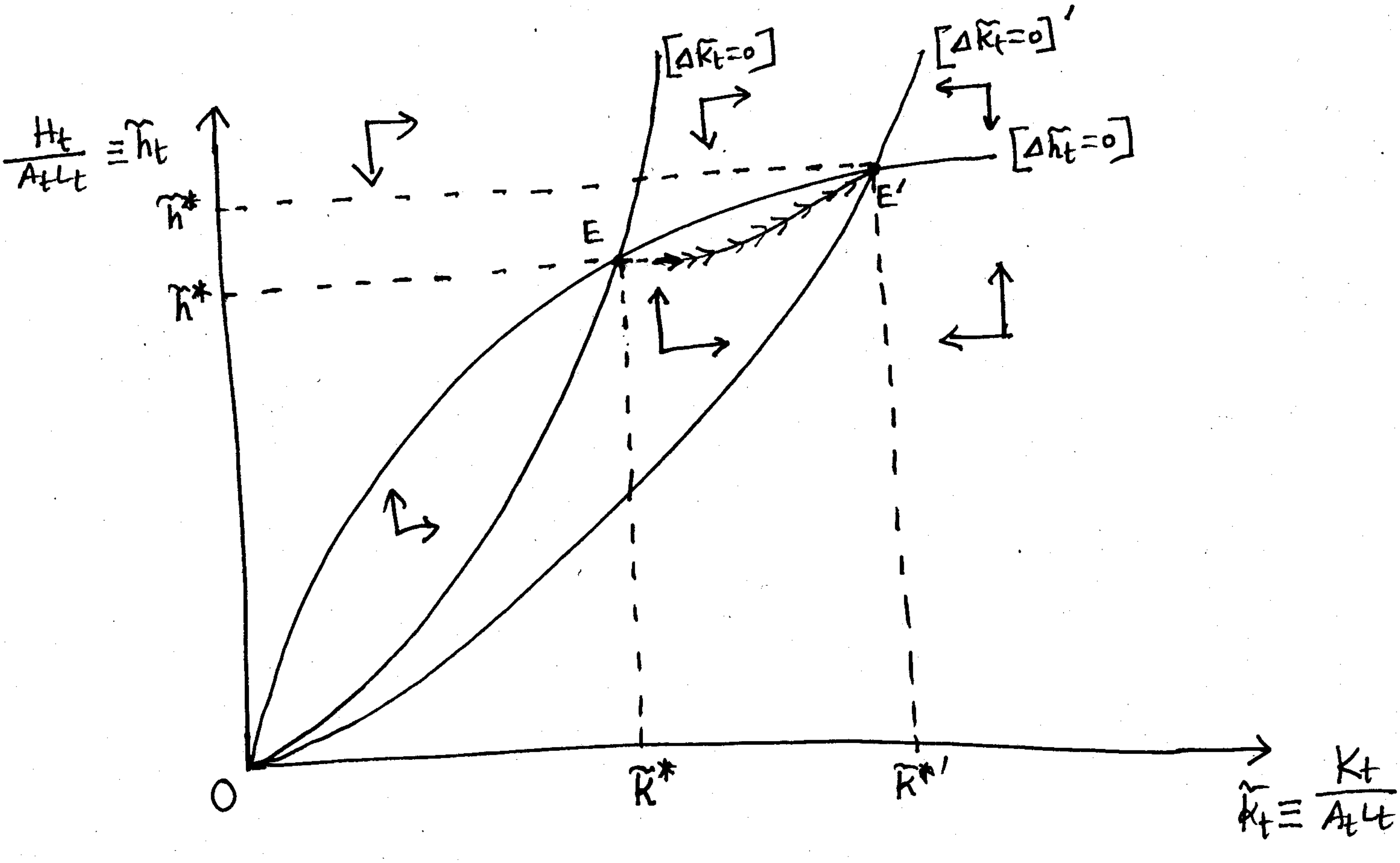

FIGURE A3(a): THE STEADY-STATE EQUILIBIRIUM OF THE BASIC SOLOW MODEL

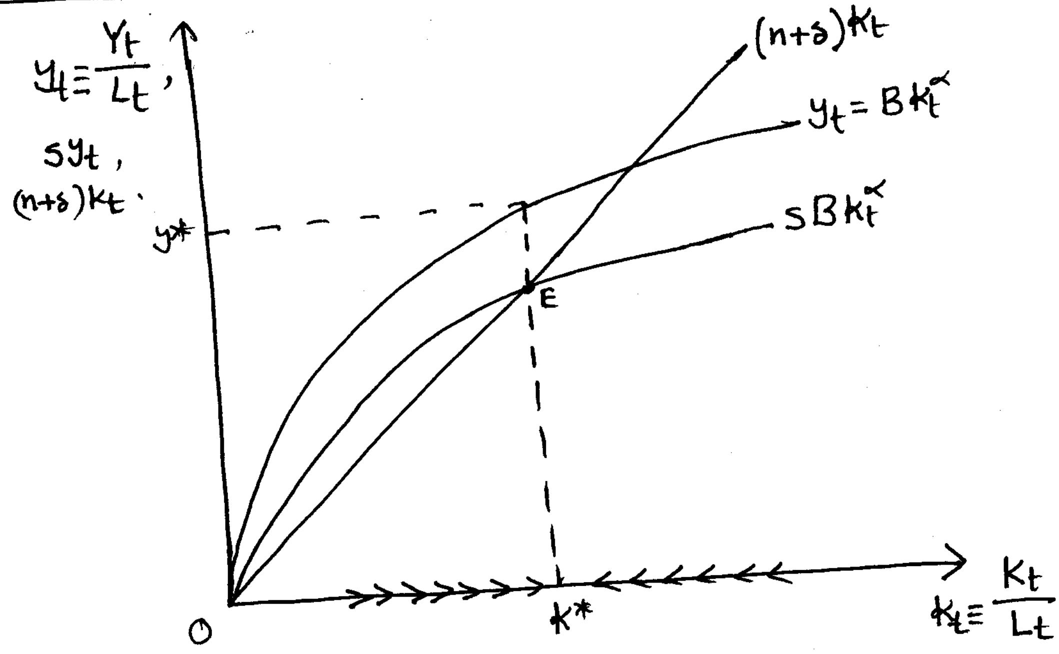

FIGURE A3(b): THE STEADY-STATE EQUILIBIRUM OF THE GENERAL SOLOW MODEL

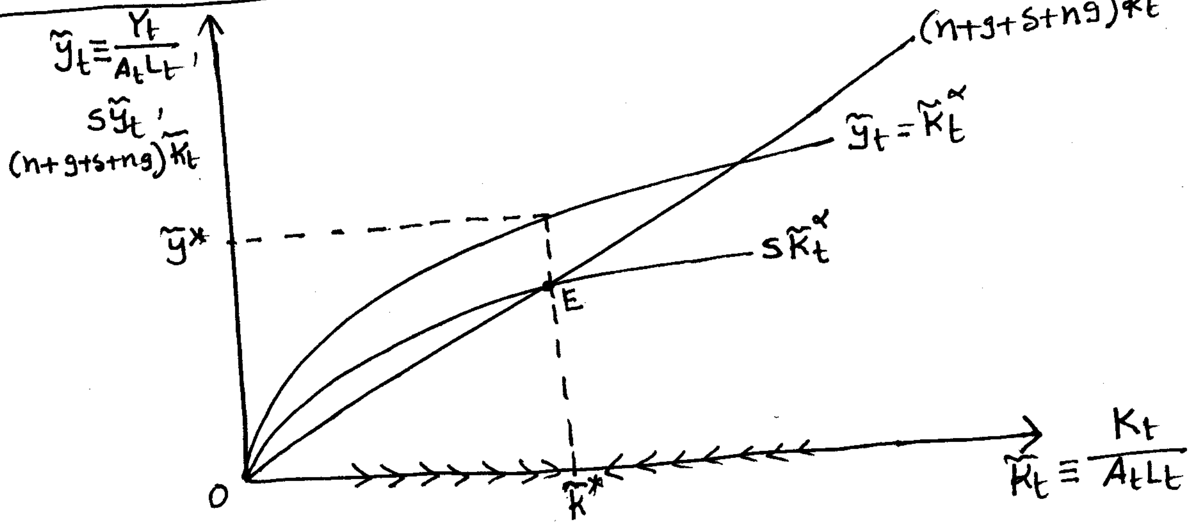

FIGURE A3(c): THE STEADY-STATE EQUILIBIRUM GROWTH PATH OF y_t IN THE GENEAL SOLOW GROWTH MODEL.

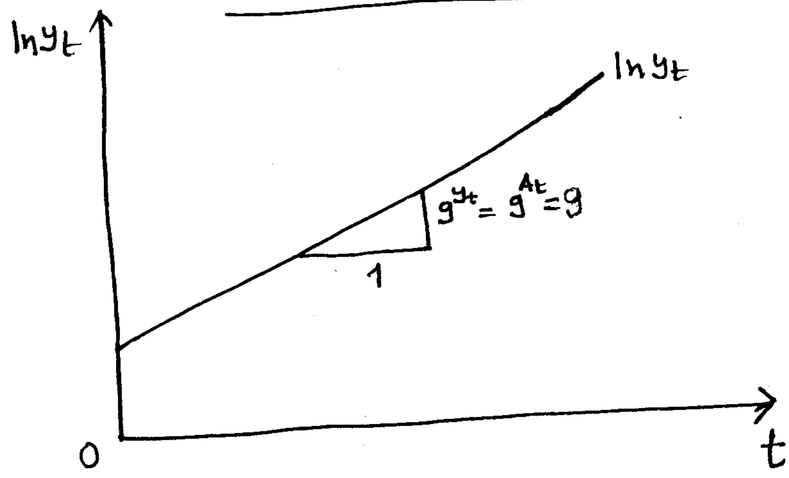

FIGURE B1(a): THE TRANSITION EQUATION AND THE STEADY-STATE EQUILIBRIUM

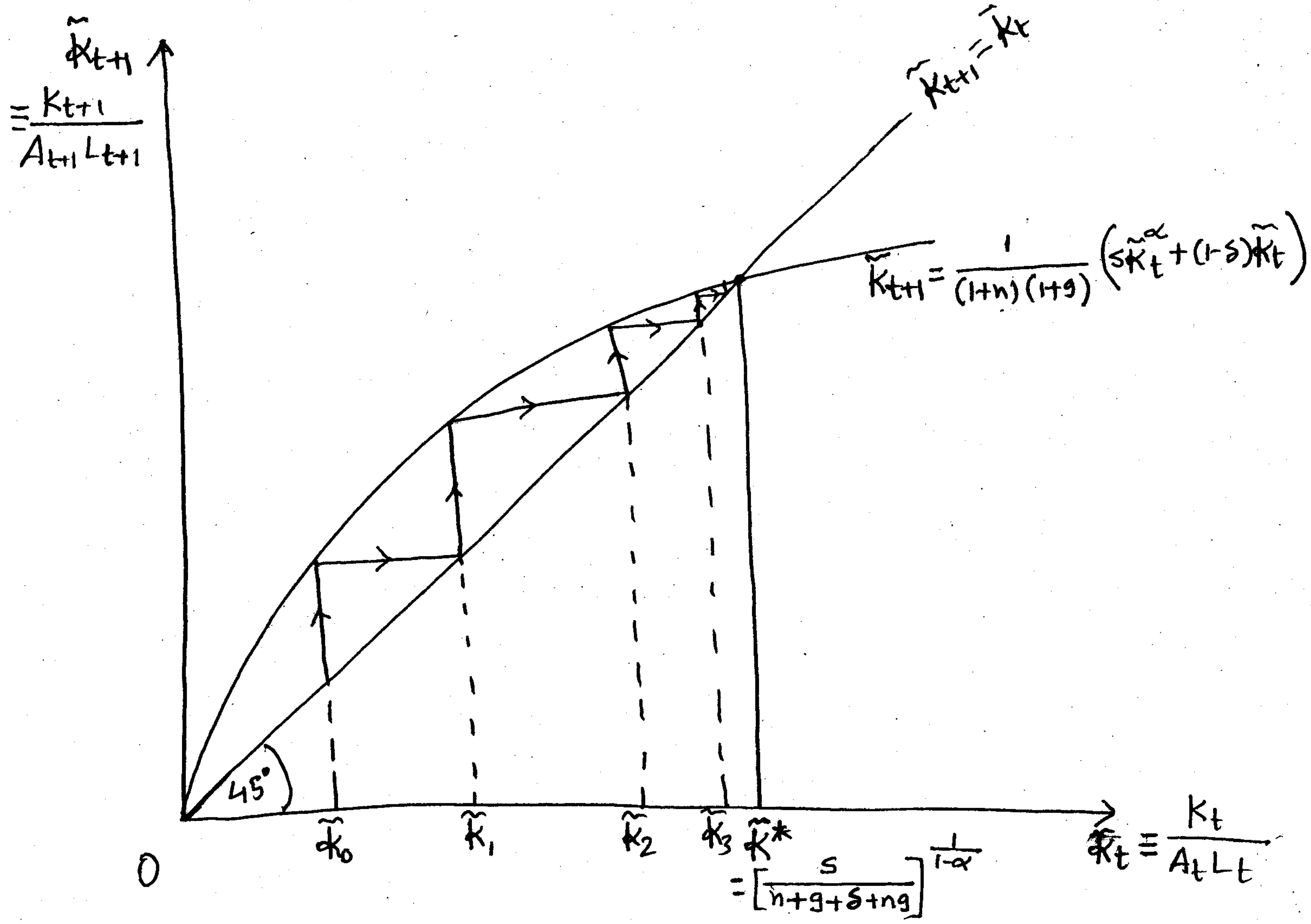

FIGURE B1(b): THE STEADY-STATE EQUILIBRIUM

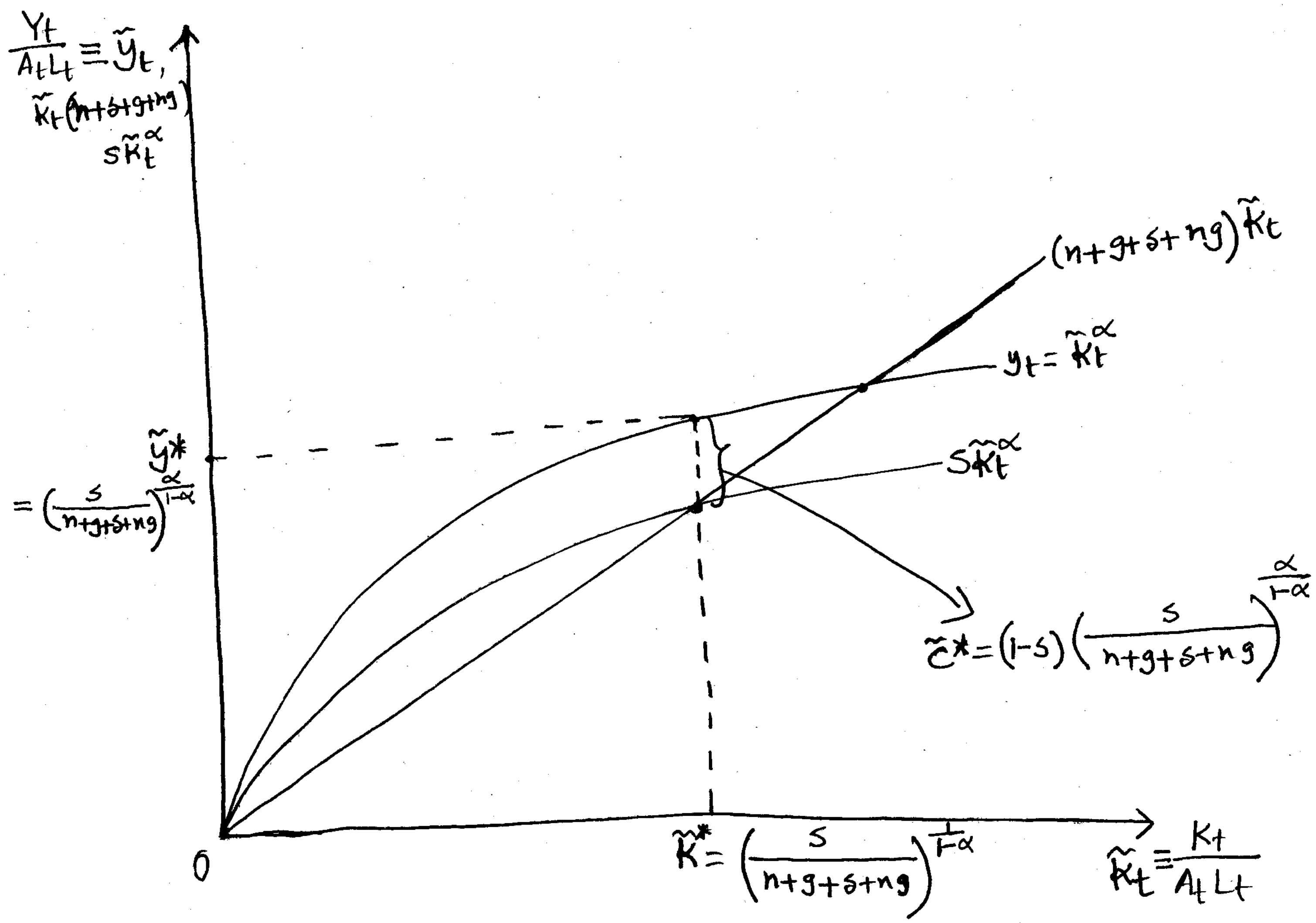

FIGURE B1(d): THE STEADY-STATE EQUILIBRIUM WITH GOLDEN RULE SAVING RATE