EC 450
Advanced Macroeconomics
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Suggested Solutions to Assignment 5 (OPTIONAL)

Part B Problem Solving Questions

Read each part of the question very carefully. Show all the steps of your calculations to get full marks.

B1.

Exercise 3 of Chapter 8 of the textbook: Part 1, 2, 3 and 4. You do NOT have to solve Part 5 of this question.

B2.

Exercise 4 of Chapter 8 of the textbook.

B3.

Exercise 5 of Chapter 8 of the textbook: Part 1 and 2.

B4.

Exercise 6 of Chapter 8 of the textbook: Part 1, 2 and 3. You do NOT have to solve Part 4 of this question.

In the special case where $\phi = 0$ we get:

$$\begin{aligned}\lambda &= (1-\alpha)\left[1-\frac{1-\delta}{1+n}\right] \\ &= (1-\alpha)\frac{1+n-(1-\delta)}{1+n} \\ &= \frac{1}{1+n}(1-\alpha)(n+\delta).\end{aligned}$$

This corresponds to (35) of Chapter 5 with $g = 0$. Setting $\phi = 0$ eliminates the productive externality and reduces the model to the general Solow model with $g = 0$ (or to the basic Solow model).

When ϕ goes to 1, the term $(1-\phi)$ appearing in the general expression for λ goes to zero, implying that the exponent $1/(1-\phi)$ goes to infinity. Under our assumption, $(1+n)^{\frac{1}{1-\phi}} > 1-\delta$, the square bracket goes to a finite value as ϕ goes to one. Hence λ goes to zero because of the presence of the factor $(1-\phi)$. A larger ϕ means a larger impact of the productive externality, reducing the degree of diminishing returns to capital (and to capital per worker), see the expression for Y_t above. It has been explained at several places, e.g. in Section 1 of this chapter, that diminishing returns to capital causes convergence, so as diminishing returns disappears convergence becomes more and more slow. When $\phi = 1$, the productive externality is strong enough to completely eliminate diminishing returns, implying no convergence at all.

Exercise 8.3: The model of semi-endogenous growth ($\phi < 1$) with the productive externality coming from K_t/L_t rather than from K_t

The learning effect may well come from 'working with computers as such', not necessarily from having one each. This speaks for the external affect arising from K_t, not from K_t/L_t. On the other hand, the learning effect may be stronger if each worker has more time with a computer, speaking for the external effect arising from K_t/L_t. None of the formulations (the external effect arising from K_t or from K_t/L_t) is therefore to be considered right or wrong. Rather these two cases cover the interesting possibilities, being the two relevant 'end points'.

1. From $A_t = (K_t/L_t)^\phi$ and $L_{t+1} = (1+n)L_t$:

$$\frac{A_{t+1}}{A_t} = \left(\frac{K_{t+1}}{L_t\,(1+n)}\right)^\phi / \left(\frac{K_t}{L_t}\right)^\phi = \frac{\left(\frac{K_{t+1}}{K_t}\right)^\phi}{(1+n)^\phi}.$$

For the transition equation proceed as in the chapter:

$$\frac{\tilde{k}_{t+1}}{\tilde{k}_t} = \frac{\frac{K_{t+1}}{K_t}}{\frac{A_{t+1}}{A_t}\frac{L_{t+1}}{L_t}} = \frac{\frac{K_{t+1}}{K_t}}{\frac{\left(\frac{K_{t+1}}{K_t}\right)^\phi}{(1+n)^\phi}\,(1+n)} = \left(\frac{1}{1+n}\right)^{1-\phi}\left(\frac{K_{t+1}}{K_t}\right)^{1-\phi}. \tag{A}$$

Inserting from (6), implying $\tilde{y}_t = \tilde{k}_t^\alpha$, and from (8) and (9) we get:

$$\begin{aligned}
\frac{\tilde{k}_{t+1}}{\tilde{k}_t} &= \left(\frac{1}{1+n}\right)^{1-\phi}\left(\frac{sY_t + (1-\delta)\,K_t}{K_t}\right)^{1-\phi} \\
&= \left(\frac{1}{1+n}\right)^{1-\phi}\left(s\frac{Y_t}{K_t} + (1-\delta)\right)^{1-\phi} \\
&= \left(\frac{1}{1+n}\right)^{1-\phi}\left(s\frac{\tilde{y}_t}{\tilde{k}_t} + (1-\delta)\right)^{1-\phi} \Leftrightarrow
\end{aligned}$$

$$\tilde{k}_{t+1} = \left(\frac{1}{1+n}\right)^{1-\phi}\tilde{k}_t\left(s\left(\tilde{k}_t\right)^{\alpha-1} + (1-\delta)\right)^{1-\phi}.$$

2. The steady state value $\tilde{k}^*$ for $\tilde{k}_t$ is found by inserting $\tilde{k}_t = \tilde{k}_{t+1} = \tilde{k}^*$ in the transition equation:

$$1 = \left(\frac{1}{1+n}\right)^{1-\phi}\left(s\left(\tilde{k}^*\right)^{\alpha-1} + (1-\delta)\right)^{1-\phi} \Leftrightarrow$$

$$1+n = s\left(\tilde{k}^*\right)^{\alpha-1} + (1-\delta) \Leftrightarrow$$

$$\tilde{k}^* = \left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}}.$$

Insert this into $\tilde{y}_t = \tilde{k}_t^\alpha$ to find that:

$$\tilde{y}^* = \left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}.$$

In steady state we have that $\tilde{k}_{t+1}/\tilde{k}_t = 1$. Insert this into the expression for $\tilde{k}_{t+1}/\tilde{k}_t$ used in Question 1 to find:

$$\left(\frac{1}{1+n}\right)^{1-\phi}\left(\frac{K_{t+1}}{K_t}\right)^{1-\phi} = 1 \Leftrightarrow$$
$$\frac{K_{t+1}}{K_t} = 1+n.$$

Inserting this into the above expression for A_{t+1}/A_t shows that $A_{t+1}/A_t = 1$. Hence there is no growth in A_t *in steady state*. Since both $\tilde{k}_t = k_t/A_t$ and A_t are constant in steady state, so must k_t be. Furthermore, Since both $\tilde{y}_t = y_t/A_t$ and A_t are constant in steady state, so must y_t be.

3. Insert $A_t = (K_t/L)^{\phi}$ into the production function, (6), to find:

$$Y_t = K_t^{\alpha}\left(\left(\frac{K_t}{L_t}\right)^{\phi} L_t\right)^{1-\alpha} = K_t^{\alpha+\phi(1-\alpha)} L_t^{(1-\alpha)(1-\phi)}.$$

Labour is productive at the aggregate level, when $\phi < 1$ (which it is not when $\phi = 1$). However, with the productive externality arising from K_t/L_t, the aggregate production function exhibits *constant* returns to K_t and L_t (since the sum of exponents, $\alpha+\phi(1-\alpha)+(1-\alpha)(1-\phi)$, equals one). As explained in the chapter, increasing returns to capital and labour in the aggregate production function are fundamental for semi-endogenous growth. Since returns to scale are no longer increasing there cannot be semi-endogenous growth.

4. Using that $k_t = \tilde{k}_t A_t$ and $A_t = k_t^{\phi}$ we find that in steady state:

$$k_t = \left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}} k_t^{\phi} \Leftrightarrow$$

$$k_t^{1-\phi} = \left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}} \Leftrightarrow$$

$$k^* = \left(\frac{s}{n+\delta}\right)^{\frac{1}{(1-\alpha)(1-\phi)}}.$$

Inserting into the per capita production function, $y_t = k_t^\alpha {A_t}^{1-\alpha}$, along with $A_t = k_t^\phi$, one gets for y^*:

$$y^* = \left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{(1-\alpha)(1-\phi)}} \left(\frac{s}{n+\delta}\right)^{\frac{\phi(1-\alpha)}{(1-\alpha)(1-\phi)}} = \left(\frac{s}{n+\delta}\right)^{\frac{\alpha+\phi(1-\alpha)}{(1-\alpha)(1-\phi)}}.$$

Hence the constant steady state level of capital per worker and of output per worker depend on ϕ: under the (realistic) assumption that $s > n + \delta$, a larger ϕ gives higher capital and income per worker in steady state. This is a natural consequence of the fact that for K_t/L_t larger than one, the aggregate economy simply becomes more productive the larger ϕ is (since $A_t = (K_t/L_t)^\phi$), and if the investment rate is sufficiently large relative to the depreciation and population growth rates, capital per worker will indeed be larger than one in steady state.

5. Linearizing the transition equation around steady state etc. (see Exercise 2 or Chapter 5) gives us the usual first order difference equation in $\tilde{y}_t$:

$$\ln \tilde{y}_{t+1} - \ln \tilde{y}_t = \lambda \left(\ln \tilde{y}^* - \ln \tilde{y}_t\right), \quad \lambda = 1 - G'\left(\tilde{k}^*\right),$$

where now:

$$G(\tilde{k}_t) = \left(\frac{1}{1+n}\right)^{1-\phi} \tilde{k}_t \left(s\left(\tilde{k}_t\right)^{\alpha-1} + (1-\delta)\right)^{1-\phi}.$$

Hence:

$$G'(\tilde{k}_t) = \left(\frac{1}{1+n}\right)^{1-\phi} \left(s\left(\tilde{k}_t\right)^{\alpha-1} + (1-\delta)\right)^{1-\phi} \left[1 + \frac{(1-\phi)\, s\, (\alpha-1)\left(\tilde{k}_t\right)^{\alpha-1}}{s\left(\tilde{k}_t\right)^{\alpha-1} + (1-\delta)}\right].$$

Now, insert the expression for $\tilde{k}^*$ in place of $\tilde{k}_t$ to get:

$$\begin{aligned} G'(\tilde{k}^*) &= \left(\frac{1}{1+n}\right)^{1-\phi} \left(s\left(\frac{s}{n+\delta}\right)^{\frac{\alpha-1}{1-\alpha}} + (1-\delta)\right)^{1-\phi} \left[1 + \frac{(1-\phi)\, s\, (\alpha-1)\left(\frac{s}{n+\delta}\right)^{\frac{\alpha-1}{1-\alpha}}}{s\left(\frac{s}{n+\delta}\right)^{\frac{\alpha-1}{1-\alpha}} + (1-\delta)}\right] \\ &= \left(\frac{1}{1+n}\right)^{1-\phi} (1+n)^{1-\phi} \left[1 - \frac{(1-\phi)(1-\alpha)(n+\delta)}{1+n}\right] \end{aligned}$$

Hence, the rate of convergence is:

$$\begin{aligned}\lambda &= 1 - G'\left(\tilde{k}^*\right) \\ &= 1 - \left(\frac{1}{1+n}\right)^{1-\phi}(1+n)^{1-\phi}\left[1 - \frac{(1-\phi)(1-\alpha)(n+\delta)}{1+n}\right].\end{aligned}$$

Insert $\phi = 1$ to find the limit of λ as ϕ goes to 1:

$$\lambda = 1 - \left(\frac{1}{1+n}\right)^0 (1+n)^0 [1-0] = 0.$$

Exercise 8.4 Balanced growth

The model of semi-endogenous growth: Since - by definition of $\tilde{k}_t$ and $\tilde{y}_t$ - we have that $k_t = \tilde{k}_t A_t$ and $y_t = \tilde{y}_t A_t$, and since $\tilde{k}_t$ and $\tilde{y}_t$ are constant in steady state, k_t and y_t must grow at the same rate as A_t in steady state, which is the rate g_{se} given by (17). Since $c_t = (1-s)\, y_t$, consumption per worker also grows at this rate. From (3) and (5) follows that $w_t L_t = (1-\alpha)Y_t \Leftrightarrow w_t = (1-\alpha)y_t$, and that $r_t K_t = \alpha Y_t \Leftrightarrow r_t = \alpha\tilde{y}_t/\tilde{k}_t$. These show, respectively, that the wage rate grows at the rate g_{se} in steady state, and that the real rate of return on capital is constant in steady state.

The model of endogenous growth: Equation (24) shows directly that the growth rate of k_t is constant and given by $g_e = sA - \delta$. Furthermore, $y_t = Ak_t$ and $c_t = (1-s)\, y_t$ imply that y_t and c_t grow at rate g_e. From (19) we can conclude that r_t is constant and that w_t grows at the same rate as K_t and k_t.

Exercise 8.5. Explosive endogenous growth

1. We can analyze this model much like we analyzed the 'AK-model' in the chapter, but, crucially, that model's A is not a constant when there is population growth. Combining (6) and (7) for $\phi = 1$ gives:

$$Y_t = K_t L_t^{1-\alpha} \Leftrightarrow y_t = k_t L_t^{1-\alpha}.$$

(In the chapter L_t was constant and we defined $A \equiv L^{1-\alpha}$ etc.).

Dividing on both sides of the capital accumulation equation (8) by L_{t+1} $(= (1+n)L_t)$ gives:

$$k_{t+1} = \frac{1}{1+n}\left(sy_t + (1-\delta)\,k_t\right),$$

and then using the expression above for y_t:

$$\begin{aligned} k_{t+1} &= \frac{1}{1+n}\left(sk_t L_t^{1-\alpha} + (1-\delta)\,k_t\right) \Leftrightarrow \\ \frac{k_{t+1}-k_t}{k_t} &= \frac{1}{1+n}\left[sL_t^{1-\alpha} - (n+\delta)\right]. \end{aligned}$$

Inserting that $L_t = L_0(1+n)^t$ gives:

$$\frac{k_{t+1}-k_t}{k_t} = \frac{1}{1+n}\left[sL_0^{1-\alpha}(1+n)^{(1-\alpha)t} - (n+\delta)\right] \equiv g_{k,t}.$$

For the growth rate of y_t use that from $y_t = k_t L_t^{1-\alpha}$ follows $y_{t+1} = k_{t+1}L_{t+1}^{1-\alpha} = k_{t+1}\left(1+n\right)^{1-\alpha} L_t^{1-\alpha}$. Combining these gives:

$$g_{y,t} \equiv \frac{y_{t+1}-y_t}{y_t} = \frac{\left(1+n\right)^{1-\alpha} k_{t+1} - k_t}{k_t}.$$

Here we can insert that $k_{t+1} = (1+g_{k,t})k_t$ giving:

$$g_{y,t} = \left(1+n\right)^{1-\alpha}\left(1+g_{k,t}\right) - 1.$$

Inserting the $g_{k,t}$ found above then gives:

$$\begin{aligned} g_{y,t} &= \left(1+n\right)^{1-\alpha}\frac{1}{1+n}\left[sL_0^{1-\alpha}(1+n)^{(1-\alpha)t} + 1 - \delta\right] - 1 \\ &= \left(\frac{1}{1+n}\right)^{\alpha}\left[sL_0^{1-\alpha}(1+n)^{(1-\alpha)t} + 1 - \delta\right] - 1. \end{aligned}$$

One can see that both $g_{k,t}$ and $g_{y,t}$ increase over time and go to infinity.

2. Inserting $A_t = K_t^{\phi}$ into the production function, $Y = K_t^{\alpha}(A_t L)^{1-\alpha}$ gives:

$$Y_t = K_t^{\alpha}(K_t^{\phi}L)^{1-\alpha} = AK_t^{1+(1-\alpha)(\phi-1)}, \;\; A \equiv L^{1-\alpha}.$$

This is inserted into (8) to find the transition equation for K_t:

$$K_{t+1} = sAK_t^{1+(1-\alpha)(\phi-1)} + (1-\delta)K_t.$$

The usual requirement for a steady state value K^* for K_t is $K_t = K_{t+1} = K^*$ yielding:

$$1 = sA\left(K^*\right)^{(1-\alpha)(\phi-1)} + 1 - \delta \Leftrightarrow$$

$$K^* = \left[\frac{\delta}{sA}\right]^{\frac{1}{(1-\alpha)(\phi-1)}},$$

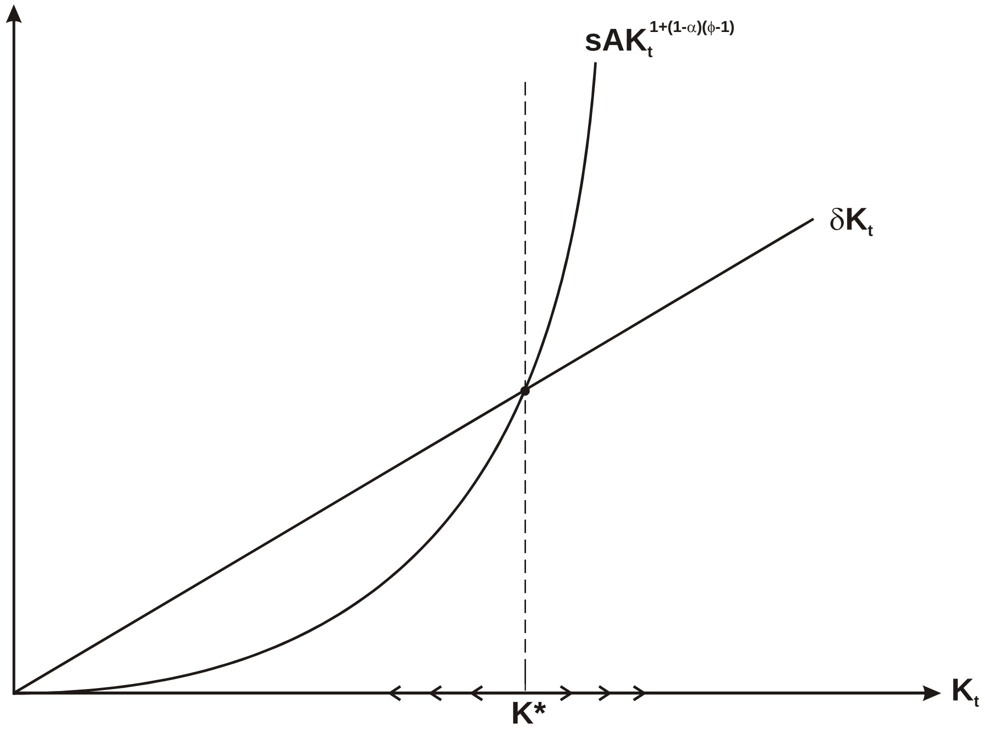
so indeed there is such a steady state. The strange feature about it is that a larger savings and investment rate or population size (remember that $A = L^{1-\alpha}$) results in a lower steady state level of capital, whereas a larger depreciation rate gives a lower more capital in steady state.

The Solow equation is:

$$K_{t+1} - K_t = sAK_t^{1+(1-\alpha)(\phi-1)} - \delta K_t.$$

The Solow diagram illustrating the Solow equation looks as shown below. Since the exponent on K_t fulfils $1 + (1-\alpha)(\phi - 1) > 1$ when $\phi > 1$, the curve $sAK_t^{1+(1-\alpha)(\phi-1)}$ looks like a x^2-function: as K_t goes to zero the slope goes to 0, and as K_t goes to infinity the slope goes to infinity.

The diagram shows that K^* is unstable, that is, if $K_t < K^*$ for some t, then K_t will decrease and converge towards zero, while if $K_t > K^*$ in period t, then K_t will increase towards infinity.

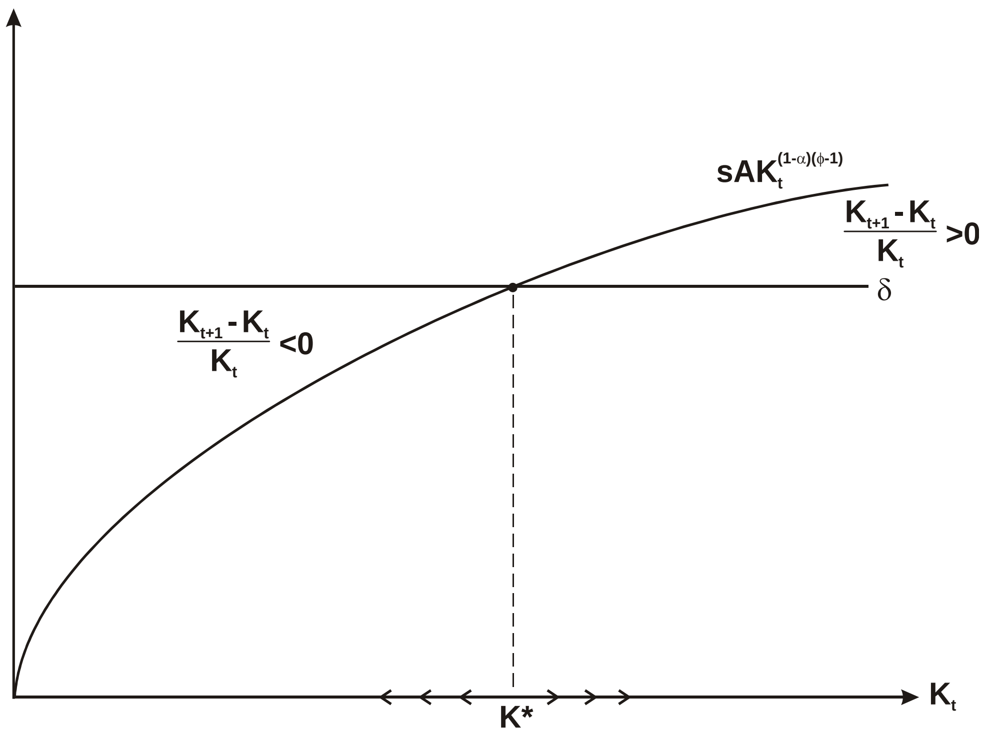


Divide the Solow equation by K_t on both sides to find the modified Solow equation:

$$\frac{K_{t+1} - K_t}{K_t} = sAK_t^{(1-\alpha)(\phi-1)} - \delta,$$

illustrated in the modified Solow diagram below. Since the exponent on K_t fulfils $0 < (1-\alpha)(\phi-1) < 1$ (remember $\phi < 2$) the curve $sAK_t^{(1-\alpha)(\phi-1)}$ looks a $\sqrt{x}$-function.

If $K_t > K^*$ the growth rate is increasing and since $sAK_t^{(1-\alpha)(\phi-1)}$ goes to infinity as K_t goes to infinity, the growth rate $(K_{t+1} - K_t)/K_t$ goes to infinity as K_t does. Moreover, when $K_t < K^*$ the growth rate of K_t is decreasing and eventually goes to zero. Since the population size is constant, $k_t \equiv K_t/L$ and its growth rate evolve similarly to K_t and the growth rate of K_t. Since $Y_t = AK_t^{1+(1-\alpha)(\phi-1)}$, similar behaviour follows for Y_t and y_t.



Exercise 8.6. Endogenous growth with both physical and human capital

See Excel spread sheet for this exercise: Chapter-08-Exc-01-Sol.

1. Insert the expression for the productive externality along with $K_t^d = K_t$, $H_t^d = H_t$, $L_t^d = L$, and $\phi = 1$ into the production function for the individual firm:

$$
\begin{aligned}
Y_t &= K_t^{\alpha} H_t^{\varphi} \left(K_t^{\frac{\alpha}{\alpha+\varphi}} H_t^{\frac{\varphi}{\alpha+\varphi}} L \right)^{1-\alpha-\varphi} \\
&= L^{1-\alpha-\varphi} K_t^{\frac{\alpha(\alpha+\varphi)+\alpha(1-\alpha-\varphi)}{\alpha+\varphi}} H_t^{\frac{\varphi(\alpha+\varphi)+\varphi(1-\alpha-\varphi)}{\alpha+\varphi}} \\
&= L^{1-\alpha-\varphi} K_t^{\frac{\alpha}{\alpha+\varphi}} H_t^{\frac{\varphi}{\alpha+\varphi}} \\
&= AK_t^{\nu} H_t^{1-\nu}, \quad A \equiv L^{1-\alpha-\varphi}, \quad \nu \equiv \frac{\alpha}{\alpha+\varphi}.
\end{aligned}
$$

A knife edge case such as $\phi = 1$ is improbable in real life, but we can view $\phi = 1$ as an approximation of a ϕ below, but close to one.

2. Using the capital accumulation equations for K_t and H_t in the definition of x_{t+1} we find that:

$$x_{t+1} \equiv \frac{K_{t+1}}{H_{t+1}} = \frac{s_K Y_t + (1-\delta) K_t}{s_H Y_t + (1-\delta) H_t}.$$

Inserting $Y_t = A K_t^{\nu} H_t^{1-\nu} = A \left(K_t / H_t\right)^{\nu} H_t = A x_t^{\nu} H_t$ gives the transition equation for x_t:

$$\begin{aligned} x_{t+1} &= \frac{s_K A x_t^{\nu} H_t + (1-\delta) K_t}{s_H A x_t^{\nu} H_t + (1-\delta) H_t} \\ &= \frac{s_K A x_t^{\nu} + (1-\delta) x_t}{s_H A x_t^{\nu} + (1-\delta)}. \end{aligned}$$

To show the existence of a steady state x^*, set $x_{t+1} = x_t = x^*$ in the transition equation:

$$x^* = \frac{s_K A (x^*)^{\nu} + (1-\delta) x^*}{s_H A (x^*)^{\nu} + (1-\delta)} \Leftrightarrow$$

$$x^* \left[s_H A (x^*)^{\nu} + (1-\delta)\right] = s_K A (x^*)^{\nu} + (1-\delta) x^* \Leftrightarrow$$

$$x^* = \frac{s_K}{s_H}.$$

This shows that there exists a unique positive point of intersection between the transition curve and the 45°-line.

The transition curve obviously passes through $(0,0)$. Convergence of x_t to x^* is therefore implied if the transition curve is everywhere increasing and its slope at zero is greater than one (a figure may be nice to look at here). Differentiate with respect to x_t:

$$\begin{aligned} \frac{\partial x_{t+1}}{\partial x_t} &= \frac{\left[\nu s_K A x_t^{\nu-1} + (1-\delta)\right]\left[s_H A x_t^{\nu} + (1-\delta)\right] - \nu s_H A x_t^{\nu-1}\left[s_K A x_t^{\nu} + (1-\delta) x_t\right]}{\left[s_H A x_t^{\nu} + (1-\delta)\right]^2} \\ &= \frac{(1-\delta)\left[s_H A x_t^{\nu} + (1-\delta)\right] + \nu s_K A x_t^{\nu-1}(1-\delta) - \nu s_H A x_t^{\nu-1}(1-\delta) x_t}{\left[s_H A x_t^{\nu} + (1-\delta)\right]^2} \\ &= \frac{(1-\delta)\left[(1-\nu) s_H A x_t^{\nu} + (1-\delta) + \nu s_K A x_t^{\nu-1}\right]}{\left[s_H A x_t^{\nu} + (1-\delta)\right]^2}. \end{aligned}$$

Obviously, $\partial x_{t+1}/\partial x_t > 0$, and furthermore $\partial x_{t+1}/\partial x_t$ goes to infinity as x_t goes to zero because of the term with $x_t^{\nu-1}$ (recall that $\nu < 1$). This shows that x_t converges to x^* from any (strictly positive) initial value of x_t.

3. From the capital accumulation equations and $Y_t = A\left(K_t/H_t\right)^{\nu} H_t$ we find for the growth rates of K_t and H_t:

$$\begin{aligned} g_t^K &\equiv \frac{K_{t+1}-K_t}{K_t} = \frac{s_K Y_t - \delta K_t}{K_t} \\ &= s_K A\left(\frac{K_t}{H_t}\right)^{\nu-1} - \delta, \end{aligned}$$

$$\begin{aligned} g_t^H &\equiv \frac{H_{t+1}-H_t}{H_t} = \frac{s_H Y_t - \delta H_t}{H_t} \\ &= s_H A\left(\frac{K_t}{H_t}\right)^{\nu} - \delta. \end{aligned}$$

Insert the steady state value $x^* = s_K/s_H$ for K_t/H_t in both of these to find the common growth rate in steady state:

$$g = g^K = g^H = s_K^{\nu} s_H^{1-\nu} A - \delta.$$

Use $Y_t = AK_t^{\nu} H_t^{1-\nu}$ to see that:

$$\frac{Y_{t+1}-Y_t}{Y_t} = \frac{Y_{t+1}}{Y_t} - 1 = \left(\frac{K_{t+1}}{K_t}\right)^{\nu}\left(\frac{H_{t+1}}{H_t}\right)^{1-\nu} - 1$$

Hence, in steady state:

$$\left(\frac{Y_{t+1}-Y_t}{Y_t}\right)^* = (1+g)^{\nu}(1+g)^{1-\nu} - 1 = g,$$

and g is also the steady state growth rate of y_t, since there is no growth in the labour force.

4. The figure below plots $g_{00,60}$ against s_H for 80 countries: