EC 450
Advanced Macroeconomics
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Suggested Solutions to Assignment 3 (OPTIONAL)

Total Marks: 50

Part A — Short Questions — [20 marks]

For the question A1 only:

Explain why the following statement is True, False, or Uncertain according to economic principles. Use diagrams and / or numerical examples where appropriate. Unsupported answers will receive no marks. It is the explanation that is important. Each question is worth 10 marks.

A1. In the Solow model with human capital, a rise in the investment rate in human capital increases the long-run level of human capital per effective worker but leaves the long-run level of physical capital per effective worker unchanged. [Diagrams required]

False

In the Solow model with human capital, a rise in the investment rate in human capital increases the long-run levels of both human capital per effective worker and physical capital per effective worker.

See Figure 6.3 and page 169 of the textbook for a graphical explanation.

A2. Consider an economy which has the following aggregate production function and human capital function:

$$Y_t = K_t^{\alpha}\left(A_t h(u_t) L_t\right)^{1-\alpha}, \qquad 0 < \alpha < 1,$$

and

$$h(u_t) = \exp(\psi u_t), \qquad \psi > 0,$$

where Y is aggregate output, K is the stock of aggregate capital, L is total labor, A is the effectiveness of labor, $h(u_t)$ is the human capital function and u is the years of education.

Derive an appropriate growth accounting formula that splits up the average annual growth rate of GDP per worker into components coming from growth in A_t, growth in capital per worker, and growth in education.

Define $y_t \equiv \frac{Y_t}{L_t}$ and $k_t \equiv \frac{K_t}{L_t}$. Thus, y_t denotes denote output per worker and k_t denote capital per worker.

Production function: $Y_t = K_t^{\alpha}\left(A_t h(u_t) L_t\right)^{1-\alpha}$ (1)

Human capital function: $h(u_t) = \exp(\psi u_t)$ (2)

Using (2) insert the expression for $h(u_t)$ into (1):

$$Y_t = K_t^{\alpha}\left(A_t \exp(\psi u_t) L_t\right)^{1-\alpha} \qquad (3)$$

Dividing both sides of equation (3) by L_t we get:

$$
\begin{aligned}
&\frac{Y_t}{L_t} = \frac{K_t^{\alpha}\left(A_t \exp(\psi u_t) L_t\right)^{1-\alpha}}{L_t} \\
&\Rightarrow \frac{Y_t}{L_t} = \frac{K_t^{\alpha}\left(A_t \exp(\psi u_t)\right)^{1-\alpha}}{L_t^{1-1+\alpha}} \\
&\Rightarrow \frac{Y_t}{L_t} = \frac{K_t^{\alpha}\left(A_t \exp(\psi u_t)\right)^{1-\alpha}}{L_t^{\alpha}} \\
&\Rightarrow \frac{Y_t}{L_t} = \left(\frac{K_t}{L_t}\right)^{\alpha}\left(A_t \exp(\psi u_t)\right)^{1-\alpha} \\
&\Rightarrow y_t = k_t^{\alpha}\left(A_t \exp(\psi u_t)\right)^{1-\alpha} \qquad (2) \qquad \text{[Using the definition of } y_t \text{ and } k_t\text{]}
\end{aligned}
$$

For another and later year $T > t$, we have:

$$y_T = k_T^{\alpha}\left(A_T \exp(\psi u_T)\right)^{1-\alpha} \qquad (3)$$

If we take logs on both sides of (2) and (3), we get:

$$\ln y_t = \alpha \ln k_t + (1-\alpha)\ln A_t + (1-\alpha)\psi u_t \qquad (4)$$
$$\ln y_T = \alpha \ln k_T + (1-\alpha)\ln A_T + (1-\alpha)\psi u_T \qquad (5)$$

Subtracting (4) from (5), and divide on both sides by *T*-*t*, we get:

$$\frac{\ln y_T - \ln y_t}{T-t} = \alpha\frac{\ln k_T - \ln k_t}{T-t} + (1-\alpha)\frac{\ln A_T - \ln A_t}{T-t} + (1-\alpha)\psi\frac{u_T - u_t}{T-t} \qquad (6)$$

Let g^{y} denote the growth rate in output per worker, g^{A} denote the growth rate of A_t and g^{k} denote the growth rate in capital per worker. Define $g^{h(u)} \equiv \psi\frac{u_T - u_t}{T-t}$. So, $g^{h(u)}$ denotes the growth in human capital or education. Using these notations, we can rewrite (6) as,

$$g^{y} = \alpha g^{k} + (1-\alpha)g^{A} + (1-\alpha)g^{h(u)} \qquad (7)$$

The growth-accounting equation (7) splits up the growth in GDP per worker into contributions from growth in capital per worker, growth in the effectiveness of labor (technological progress) and growth in education.

Part B **Problem Solving Questions** **[30 marks]**

Read each part of the question very carefully. Show all the steps of your calculations to get full marks.

B1. [30 Marks]

Consider the Solow model with human capital. Assume that the economy has a Cobb-Douglas aggregate production function with labor-augmenting technological progress:

$$Y_t = K_t^{\lambda} H_t^{\varphi} (A_t N_t)^{1-\lambda-\varphi}, \qquad 0 < \lambda < 1, \qquad 0 < \varphi < 1, \qquad \lambda + \varphi < 1,$$

where *Y* is aggregate output, *K* is the stock of aggregate physical capital, *H* the stock of aggregate capital, *N* is total labor and *A* is the effectiveness of labor. Assume that *L* and *A* grow exogenously at constant rates *n* and *g*, respectively. Capital depreciates at a constant rate δ.

In this economy, hiring one more (marginal) unit of labor means hiring one more unit endowed with the average amount, h_t, of human capital per worker. Hence, a firm cannot increase the input of 'raw' labor, N_t, without increasing proportionally the input of human capital, $H_t = h_t L_t$.

Denote with lower case letters the variables in unit of effective worker. That means,

$$\tilde{y} \equiv \frac{Y}{AL}, \ \tilde{k} \equiv \frac{K}{AL} \text{ and } \tilde{h} \equiv \frac{H}{AL}.$$

The evolution of aggregate physical capital in the economy is given by

$$K_{t+1} - K_t = s_K Y_t - \delta K_t$$

where s_K is a constant and exogenous physical capital investment rate.

Similarly, the evolution of aggregate human capital in the economy is given by

$$H_{t+1} - H_t = s_H Y_t - \delta H_t$$

where s_H is a constant and exogenous human capital investment rate.

(a) Derive the laws of motion, or the transition equations, for physical capital per effective worker and human capital per effective worker. [4 marks]

Define $\tilde{y}_t \equiv \frac{Y_t}{A_t N_t}$, $\tilde{k}_t \equiv \frac{K_t}{A_t N_t}$ and $\tilde{h}_t \equiv \frac{H_t}{A_t L_t}$. Thus, $\tilde{y}_t$ denotes output per effective worker, $\tilde{k}_t$ denotes physical capital per effective worker and $\tilde{h}_t$ denotes human capital per effective worker

Production function: $Y_t = K_t^{\lambda} H_t^{\varphi} (A_t N_t)^{1-\lambda-\varphi}$ (1)

An exogenous growth in labor: $N_{t+1} = (1+n)N_t$ (2)

An exogenous growth in the effectiveness of labor : $A_{t+1} = (1+g)A_t$ (3)

The evolution equation of aggregate capital: $K_{t+1} - K_t = sY_t - \delta K_t$ (4)

The evolution of aggregate human capital: $H_{t+1} - H_t = s_H Y_t - \delta H_t$ (5)

Dividing both sides of the production function, equation (1), by $A_t N_t$ we get:

$$\frac{Y_t}{A_t N_t} = \frac{K_t^{\lambda} H_t^{\varphi} (A_t N_t)^{1-\lambda-\varphi}}{A_t N_t}$$

$$\Rightarrow \frac{Y_t}{A_t N_t} = \frac{K_t^{\lambda} H_t^{\varphi}}{(A_t N_t)^{1-1+\lambda+\varphi}}$$

$$\Rightarrow \frac{Y_t}{A_t N_t} = \frac{K_t^{\lambda} H_t^{\varphi}}{(A_t N_t)^{\lambda+\varphi}}$$

$$\Rightarrow \frac{Y_t}{A_t N_t} = \left(\frac{K_t}{A_t N_t}\right)^{\lambda} \left(\frac{H_t}{A_t N_t}\right)^{\varphi}$$

$$\Rightarrow \tilde{y}_t = \tilde{k}_t^{\lambda} \tilde{h}_t^{\varphi} \quad (6)$$

[Using the definitions of $\tilde{y}_t$, $\tilde{k}_t$ and $\tilde{h}_t$.]

Rewrite equation (4) as:

$$K_{t+1} = s_K Y_t + (1-\delta)K_t. \quad (4.1)$$

Dividing both sides of equation (4.1) by $A_{t+1} N_{t+1}$ we get:

$$\frac{K_{t+1}}{A_{t+1} N_{t+1}} = \frac{s_K Y_t + (1-\delta)K_t}{A_{t+1} N_{t+1}}$$

$$\Rightarrow \frac{K_{t+1}}{A_{t+1} N_{t+1}} = \frac{s_K Y_t + (1-\delta)K_t}{(1+g)A_t (1+n)N_t}$$

[Using equations (2) and (3)]

$$\Rightarrow \frac{K_{t+1}}{A_{t+1}N_{t+1}} = \frac{1}{(1+n)(1+g)}\left(s_K \frac{Y_t}{A_t N_t} + (1-\delta)\frac{K_t}{A_t N_t} \right)$$

$$\Rightarrow \tilde{k}_{t+1} = \frac{1}{(1+n)(1+g)}\left(s_K \tilde{y}_t + (1-\delta)\tilde{k}_t \right) \qquad \text{[Using the definitions of } \tilde{y}_t \text{ and } \tilde{k}_t\text{]}$$

$$\Rightarrow \tilde{k}_{t+1} = \frac{1}{(1+n)(1+g)}\left(s_K \tilde{k}_t^{\lambda} \tilde{h}_t^{\varphi} + (1-\delta)\tilde{k}_t \right) \quad \text{[Substituting } \tilde{y}_t \text{ using (6)]}$$

Therefore, the transition equation for physical capital per effective worker is:

$$\tilde{k}_{t+1} = \frac{1}{(1+n)(1+g)}\left(s_K \tilde{k}_t^{\lambda} \tilde{h}_t^{\varphi} + (1-\delta)\tilde{k}_t \right) \qquad (7)$$

Rewrite equation (5) as:

$$H_{t+1} = s_H Y_t + (1-\delta)H_t\,. \qquad (5.1)$$

Dividing both sides of equation (5.1) by $A_{t+1}N_{t+1}$ we get:

$$\frac{H_{t+1}}{A_{t+1}N_{t+1}} = \frac{s_H Y_t + (1-\delta)H_t}{A_{t+1}N_{t+1}}$$

$$\Rightarrow \frac{H_{t+1}}{A_{t+1}N_{t+1}} = \frac{s_H Y_t + (1-\delta)H_t}{(1+g)A_t(1+n)N_t} \qquad \text{[Using equations (2) and (3)]}$$

$$\Rightarrow \frac{H_{t+1}}{A_{t+1}N_{t+1}} = \frac{1}{(1+n)(1+g)}\left(s_H \frac{Y_t}{A_t N_t} + (1-\delta)\frac{H_t}{A_t N_t} \right)$$

$$\Rightarrow \tilde{h}_{t+1} = \frac{1}{(1+n)(1+g)}\left(s_H \tilde{y}_t + (1-\delta)\tilde{h}_t \right) \qquad \text{[Using the definitions of } \tilde{y}_t \text{ and } \tilde{h}_t\text{]}$$

$$\Rightarrow \tilde{h}_{t+1} = \frac{1}{(1+n)(1+g)}\left(s_H \tilde{k}_t^{\lambda} \tilde{h}_t^{\varphi} + (1-\delta)\tilde{h}_t \right) \quad \text{[Substituting } \tilde{y}_t \text{ using (6)]}$$

Therefore, the transition equation for human capital per effective worker is:

$$\tilde{h}_{t+1} = \frac{1}{(1+n)(1+g)}\left(s_H \tilde{k}_t^{\lambda} \tilde{h}_t^{\varphi} + (1-\delta)\tilde{h}_t \right) \qquad (8)$$

(b) Solve for the steady state equilibrium values of physical capital per effective worker, human capital per effective worker and output per effective worker. [6 marks]

For the computation of steady state and for comparative static analysis it is convenient first to rewrite (7) and (8) as 'Solow equations' by subtracting $\tilde{k}_t$ from both sides of (7) and $\tilde{h}_t$ from both sides of (8):

$$\tilde{k}_{t+1} - \tilde{k}_t = \frac{1}{(1+n)(1+g)}\left(s_K \tilde{k}_t^{\lambda} \tilde{h}_t^{\varphi} + (1-\delta)\tilde{k}_t\right) - \tilde{k}_t$$

$$\Rightarrow \tilde{k}_{t+1} - \tilde{k}_t = \frac{1}{(1+n)(1+g)}\left(s_K \tilde{k}_t^{\lambda} \tilde{h}_t^{\varphi} + (1-\delta)\tilde{k}_t - (1+n+g+ng)\tilde{k}_t\right)$$

$$\therefore \tilde{k}_{t+1} - \tilde{k}_t = \frac{1}{(1+n)(1+g)}\left(s_K \tilde{k}_t^{\lambda} \tilde{h}_t^{\varphi} - (n+g+\delta+ng)\tilde{k}_t\right) \qquad (9)$$

$$\tilde{h}_{t+1} - \tilde{h}_t = \frac{1}{(1+n)(1+g)}\left(s_H \tilde{k}_t^{\lambda} \tilde{h}_t^{\varphi} + (1-\delta)\tilde{h}_t\right) - \tilde{h}_t$$

$$\Rightarrow \tilde{h}_{t+1} - \tilde{h}_t = \frac{1}{(1+n)(1+g)}\left(s_H \tilde{k}_t^{\lambda} \tilde{h}_t^{\varphi} + (1-\delta)\tilde{h}_t - (1+n+g+ng)\tilde{h}_t\right)$$

$$\therefore \tilde{h}_{t+1} - \tilde{h}_t = \frac{1}{(1+n)(1+g)}\left(s_H \tilde{k}_t^{\lambda} \tilde{h}_t^{\varphi} - (n+g+\delta+ng)\tilde{h}_t\right) \qquad (10)$$

At the steady-state equilibrium $\tilde{k}_{t+1} = \tilde{k}_t$ and $\tilde{h}_{t+1} = \tilde{h}_t$. So, a steady-state equilibrium is then a positive solution in $\tilde{k}_t$ and $\tilde{h}_t$ to the equations that appear by setting the left-sides of (9) and (10) equal to zero:

$$s_K \tilde{k}_t^{\lambda} \tilde{h}_t^{\varphi} - (n+g+\delta+ng)\tilde{k}_t = 0 \qquad (11)$$

$$s_H \tilde{k}_t^{\lambda} \tilde{h}_t^{\varphi} - (n+g+\delta+ng)\tilde{h}_t = 0 \qquad (12)$$

Solving (11) and (12) simultaneously we find the steady-state equilibrium values of physical capital per effective worker, $\tilde{k}^*$, and human capital per effective worker, $\tilde{h}^*$ (see class notes for intermediate steps):

$$\tilde{k}^* = \left(\frac{s_K^{1-\varphi} s_H^{\varphi}}{n+g+\delta+ng}\right)^{\frac{1}{1-\lambda-\varphi}} \qquad (13)$$

$$\tilde{h}^* = \left(\frac{s_K^{\lambda} s_H^{1-\lambda}}{n + g + \delta + ng} \right)^{\frac{1}{1-\lambda-\varphi}} \qquad (14)$$

Using (6) the steady-state equilibrium value for output per effective worker, $\tilde{y}^*$, is:

$$\tilde{y}^* = \left(\tilde{k}^*\right)^{\lambda} \left(\tilde{h}^*\right)^{\varphi}$$

$$\Rightarrow \tilde{y}^* = \left(\frac{s_K^{1-\varphi} s_H^{\varphi}}{n + g + \delta + ng} \right)^{\frac{\lambda}{1-\lambda-\varphi}} \left(\frac{s_K^{\lambda} s_H^{1-\lambda}}{n + g + \delta + ng} \right)^{\frac{\varphi}{1-\lambda-\varphi}} \text{ [using (13) and (14)]}$$

$$\Rightarrow \tilde{y}^* = \left(\frac{s_K}{n + g + \delta + ng} \right)^{\frac{\lambda}{1-\lambda-\varphi}} \left(\frac{s_H}{n + g + \delta + ng} \right)^{\frac{\varphi}{1-\lambda-\varphi}} \qquad (15)$$

(c) Draw a phase diagram and illustrate the steady-state equilibrium values of physical capital per effective worker and human capital per effective worker. Explain how the economy converges to the steady-state equilibrium starting from a given set of initial positive levels of physical capital per effective worker and human capital per effective worker. [6 marks]

See Figure 6.2 (page 168) of the textbook for the phase diagram. See pages 168-169 of the textbook and class notes for the explanation.

(d) Find the growth rates of physical capital per effective worker, human capital per effective worker, output per effective worker, output per worker, physical capital per worker, human capital per worker, aggregate output, aggregate capital and aggregate human capital on the balanced growth path. [9 marks]

On the balanced growth path (steady-state equilibrium),

Growth rate of physical capital per effective worker = 0

Growth rate of human capital per effective worker = 0

Growth rate of output per effective worker = 0

Growth rate of human capital per worker = g

Growth rate of physical capital per worker = g

Growth rate of output per worker = g

Growth rate of aggregate human capital = $n+g$

Growth rate of aggregate physical capital = $n+g$

Growth rate of aggregate output = $n+g$

[See the solution to B1(c) of Assignment 2 for the intermediate steps]

(e) Find the steady-state equilibrium value of output per worker. Find also the elasticity of the steady-state equilibrium output per worker with respect to the physical capital investment rate. Compare this elasticity with its counterpart in the general Solow model. Which elasticity is larger? Explain why? [5 marks]

Using the definition of $\tilde{y}_t$, the steady state equilibrium value of output per worker, y_t^*, is:

$$y_t^* = A_t \tilde{y}^*$$

$$\therefore y_t^* = A_t \left(\frac{s_K}{n + g + \delta + ng} \right)^{\frac{\lambda}{1-\lambda-\varphi}} \left(\frac{s_H}{n + g + \delta + ng} \right)^{\frac{\varphi}{1-\lambda-\varphi}}. \qquad (16)$$

[using equation (15)]

Taking logs on both sides of (16) gives:

$$\ln y_t^* = \ln A_t + \frac{\lambda}{1-\lambda-\varphi} [\ln s_K - \ln(n + g + \delta + ng)] + \frac{\varphi}{1-\lambda-\varphi} [\ln s_H - \ln(n + g + \delta + ng)]$$

(17)

The elasticity of y_t^* with respect to the physical capital investment rate, s_K $\equiv \frac{d \ln y_t^*}{d \ln s} = \frac{\lambda}{1-\lambda-\varphi}$.

See pages 170 of the textbook for an answer to the rest of the parts of Question **(e)**.