EC 450
Advanced Macroeconomics
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Suggested Solutions to Assignment 1 (REQUIRED)

Total Marks: 50

Part A True/ False/ Uncertain Questions [20 marks]

Explain why the following statement is True, False, or Uncertain according to economic principles. Use diagrams and / or numerical examples where appropriate. Unsupported answers will receive no marks. It is the explanation that is important. Each question is worth 10 marks.

A1.

During the long periods of relatively constant growth in GDP per worker in the typical Western economy, the capital-labor ratio has stayed relatively constant.

False

During the long periods of relatively constant growth in GDP per worker in the typical Western economy, the capital-labor ratio has grown by approximately the same rate as GDP per worker.

It is a stylized fact that during the long periods of relatively constant growth in GDP per worker in the typical Western economy, labor's share of GDP has stayed relatively constant. This fact implies that the share of all other production factors must also has stayed relatively constant (since this latter share is one minus labor's share). Let us call the other factors 'capital'.

Let the total capital input in year *t* be denoted by K_t. If we denote the rate of return on capital by r_t, then capital's share is $\frac{r_t K_t}{Y_t} = \frac{r_t}{\left(\frac{Y_t}{K_t}\right)}$. Hence, constancy of capital's share implies that the real rate of return on capital, r_t, and the output-capital ratio, $\frac{Y_t}{K_t}$, must have been changing by the same rates.

The long-run data in the typical Western economy supports the claim that the real rate of return on capital has been relatively constant. If capital's share, $\frac{r_t}{\left(\frac{Y_t}{K_t}\right)}$, and the rate of return on capital, r_t, have both been relatively constant, then the output-capital ratio, $\frac{Y_t}{K_t}$, and the capital-output ratio, $\frac{K_t}{Y_t}$, must also been constant. We can rewrite the capital-output ratio as $\frac{K_t}{Y_t} = \frac{\left(K_t/L_t\right)}{Y_t/L_t} = \frac{k_t}{y_t}$, where we have denoted the capital-labour ratio, or capital intensity, K_t/L_t, by k_t and denoted GDP per worker, Y_t/L_t, by y_t. Constancy of $\frac{K_t}{Y_t}$ implies that the capital-labor ratio must have been growing at the same rate as GDP per worker.

A2.

In the basic Solow model, an increase in the level of total factor productivity raises the long-run growth rate of aggregate output per worker. [Diagrams required]

False

In the basic Solow model with no technological progress, an increase in the level of total factor productivity will have no effect on the long-run growth rate of aggregate output per worker.

Figure A2 shows the effects of an increase in the level of total factor productivity on the long-run growth rate of aggregate output per worker. Assume that the economy was initially at a steady-state equilibrium at point *E* with the level of capital per worker (the capital-labor ratio or the capital intensity) at k_0^* and the level of output per worker at y_0^*. At this long-run equilibrium actual savings, sBk_t^{α}, equals the break-even investment, $(n+\delta)k_t$, which is required to keep the level of capital per worker unchanged at k_0^*.

The initial steady-state capital-labor ratio is constant at $k_0^* = B^{\frac{1}{1-\alpha}}\left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}}$ and the initial steady-state output per worker is constant at $y_0^* = B^{\frac{1}{1-\alpha}}\left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$. That is, initially the long-run growth rates in capital per worker and output per worker were both zero.

Assume that in year zero the level of total factor productivity increases from B to B' and stays at its new higher level thereafter, while no other parameters of the model change. In year zero the initial capital intensity will remain unchanged at k_0^*, because it is pre-determined and given by the capital accumulation and population dynamics in the past, where the level of total factor productivity was B. But in year zero output per worker will increase from $B(k_0^*)^\alpha$ to $B'(k_0^*)^\alpha$ because of the higher level of total factor productivity. The increased level of output per worker will increase the level of actual savings from $sB(k_0^*)^\alpha$ to $sB'(k_0^*)^\alpha$. These two effects on output per worker and actual savings per worker are shown by upwards shifts in the output per worker curve and actual savings per worker curve, respectively in Figure A2.

In year zero, the actual savings will therefore be higher than the level of break-even which is required to keep the level of capital per worker unchanged at k_0^*. In Figure A2, $sB'(k_0^*)^\alpha > (n+\delta)k_0^*$, so capital per worker will increase from year zero to year one. With $k_1 > k_0^*$, the output per worker will increase even further in year one, $B'(k_1)^\alpha > B'(k_0^*)^\alpha$, and out of the increased income per worker will come even more savings per worker, $sB'(k_1)^\alpha > sB'(k_0^*)^\alpha$. In Figure A2, $sB'(k_1)^\alpha > (n+\delta)k_1$, so capital per worker will increase again from year one to year two, $k_2 > k_1$, and so on . In the long run successive increases in the capital intensity and in output per worker will make k_t converge to the new and high steady state value $k_F^* = (B')^{\frac{1}{1-\alpha}}\left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}}$, which is again a constant value and y_t will converge to the new and high steady state value $y_F^* = (B')^{\frac{1}{1-\alpha}}\left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$, which is also a constant value. The point *F* in Figure A2 shows this new steady state equilibrium.

Thus, during the transition period from point *E* to *F*, there will be a positive growth rate in the capital intensity and in the output per worker. But once the economy reaches the new long-run equilibrium at *F*, there will be no further positive growth in the capital intensity and in the output per worker. In year zero after the increase in the level of total factor productivity, there will be a jump in the growth rate of output per worker from zero to a positive value. During the transition period, the growth rate of output per worker will gradually decrease and will become again zero once the economy reaches the new steady state equilibrium at *F*. Therefore, an increase in the level of total factor productivity has no influence on the long-run growth rate of aggregate output per worker. It only increases the growth rate of aggregate output per worker for the transition period. It, however, increases the long-run level of aggregate output per worker.

Part B Problem Solving Questions [30 marks]

Read each part of the question very carefully. Show all the steps of your calculations to get full marks.

B1. [30 Marks]

Consider a Solow economy with no technological progress. Assume that the production function is Cobb-Douglas:

$$Y_t = AK_t^{\lambda} N_t^{1-\lambda}, \qquad 0 < \lambda < 1,$$

where *Y* is aggregate output, *K* is the stock of aggregate capital, *N* is total labor and *A* is the total factor productivity. Assume that *N* grows exogenously at a constant rate *n*. Capital depreciates at a constant rate δ. There is no technological progress. The evolution of aggregate capital in the economy is given by

$$K_{t+1} - K_t = sY_t - \delta K_t$$

where s is a constant and exogenous saving rate.

(a) Derive the basic law of motion, or the transition equation, for the aggregate capital per unit of labor. Plot the transition equation in a diagram. Explain how the level of aggregate capital per unit of labor will converge to the steady state value from a given positive initial value. [8 marks]

Define $y_t \equiv \dfrac{Y_t}{N_t}$ and $k_t \equiv \dfrac{K_t}{N_t}$. Thus, y_t denotes denote output per worker and k_t denote capital per worker.

Production function: $Y_t = AK_t^{\lambda} N_t^{1-\lambda}$ (1)

An exogenous growth in labor: $N_{t+1} = (1+n)N_t$ (2)

The evolution equation of aggregate capital: $K_{t+1} - K_t = sY_t - \delta K_t$ (3)

Dividing both sides of the production function, equation (1), by N_t we get:

$$\frac{Y_t}{N_t} = \frac{AK_t^{\lambda} N_t^{1-\lambda}}{N_t}$$

$$\Rightarrow \frac{Y_t}{N_t} = \frac{AK_t^{\lambda}}{N_t^{1-1+\lambda}}$$

$$\Rightarrow \frac{Y_t}{N_t} = \frac{AK_t^{\lambda}}{N_t^{\lambda}}$$

$$\Rightarrow \frac{Y_t}{N_t} = A\left(\frac{K_t}{N_t}\right)^{\lambda}$$

$$\Rightarrow y_t = Ak_t^{\lambda} \qquad (4) \qquad \text{[Using the definition of } y_t \text{ and } k_t\text{]}$$

Rewrite equation (3) as,

$$K_{t+1} = sY_t + (1-\delta)K_t \qquad (3.1)$$

Dividing both sides of equation (3.1) by N_{t+1} we get:

$$\frac{K_{t+1}}{N_{t+1}} = \frac{sY_t + (1-\delta)K_t}{N_{t+1}}$$

$$\Rightarrow \frac{K_{t+1}}{N_{t+1}} = \frac{sY_t + (1-\delta)K_t}{(1+n)N_t} \qquad \text{[Using equation (2)]}$$

$$\Rightarrow \frac{K_{t+1}}{N_{t+1}} = \frac{1}{1+n}\left(s\frac{Y_t}{N_t} + (1-\delta)\frac{K_t}{N_t}\right)$$

$$\Rightarrow k_{t+1} = \frac{1}{1+n}\left(sy_t + (1-\delta)k_t\right) \qquad \text{[Using the definition of } y_t \text{ and } k_t\text{]}$$

$$\Rightarrow k_{t+1} = \frac{1}{1+n}\left(sAk_t^{\lambda} + (1-\delta)k_t\right) \qquad \text{[Substituting } y_t \text{ using (4)]}$$

Therefore, the transition equation for the capital per unit of labor is:

$$k_{t+1} = \frac{1}{1+n}\left(sAk_t^{\lambda} + (1-\delta)k_t\right) \qquad (5)$$

Figure B1(a) shows the transition equation as given by (5). This curve starts at (0,0) and is everywhere increasing. The 45^0 line, $k_{t+1} = k_t$, has also been drawn.

Differentiating (5) gives:

$$\frac{dk_{t+1}}{dk_t} = \frac{s\lambda A k_t^{\lambda-1} + (1-\delta)}{1+n}.$$

This shows that the slope of the transition curve decreases monotonically from infinity to $(1-\delta)/(1+n)$, as k_t increases from zero to infinity. The latter slope is positive and less than one if $n+\delta > 0$, or $n > -\delta$, which is plausible empirically. Hence the transition curve must have a unique intersection with the 45^0 line (which has slope one), to the right of $k_t = 0$.

Assume an initial level of capital per worker $k_0 > 0$, as drawn in Figure B1(a). Then k_1 will be the vertical distance from the horizontal axis up to the transition curve, and by going from the associated point on the transition curve horizontally to the 45^0 line, and then vertically down, k_1 will be taken to the horizontal axis as shown. Now, k_2 will be the vertical distance up to the transition curve. The dynamic evolution of the capital intensity (capital per worker) is given by the staircase broken line. It follows that over time, k_t will converge to the specific value given by the intersection between the transition curve and the 45^0 line, and it will do so monotonically, getting closer and closer all the time and never transcending to the other side of the intersection point. Furthermore, the convergence is global in the sense that it holds for any strictly positive initial k_0.

The intersection between the transition curve and the 45^0 line is the unique positive solution, k^*, which is obtained by setting $k_{t+1} = k_t = k$ in (5) and solving for k. This k^* is called the steady state value of aggregate capital per worker.

(b) Solve for the steady state equilibrium values of aggregate capital per worker, aggregate output per worker, consumption per worker, the real rental rate for capital and the real wage rate for labor. Illustrate the steady-state equilibrium values of aggregate capital per worker, aggregate output per worker and consumption per worker in a diagram. [8 marks]

To solve for the steady state equilibrium value of aggregate capital per worker we set $k_{t+1} = k_t = k$ in (5),

$$k = \frac{1}{1+n}\left(sAk^{\lambda} + (1-\delta)k\right)$$
$$\Rightarrow k(1+n) = sAk^{\lambda} + (1-\delta)k$$
$$\Rightarrow k + nk = sAk^{\lambda} + k - \delta k$$
$$\Rightarrow k + nk - k + \delta k = sAk^{\lambda}$$
$$\Rightarrow (n+\delta)k = sAk^{\lambda}$$

$$\Rightarrow \frac{k}{k^{\lambda}} = \frac{sA}{n+\delta}$$
$$\Rightarrow k^{(1-\lambda)} = A\left(\frac{s}{n+\delta}\right)$$
$$\Rightarrow \left(k^{(1-\lambda)}\right)^{\frac{1}{1-\lambda}} = A^{\frac{1}{1-\lambda}}\left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\lambda}}$$
$$\therefore k^{*} = A^{\frac{1}{1-\lambda}}\left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\lambda}} \qquad (6)$$

Therefore, the steady state equilibrium value of aggregate capital per worker is $A^{\frac{1}{1-\lambda}}\left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\lambda}}$.

The steady state equilibrium value of aggregate output per worker, y^*, is:

$$y^* = Ak^{*\lambda}$$

$$\Rightarrow y^* = A\left[A^{\frac{1}{1-\lambda}}\left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\lambda}}\right]^{\lambda}$$

$$\Rightarrow y^* = A\left(A^{\frac{\lambda}{1-\lambda}}\right)\left(\frac{s}{n+\delta}\right)^{\frac{\lambda}{1-\lambda}}$$

$$\Rightarrow y^* = A^{1+\frac{\lambda}{1-\lambda}}\left(\frac{s}{n+\delta}\right)^{\frac{\lambda}{1-\lambda}}$$

$$\Rightarrow y^* = A^{\frac{1-\lambda+\lambda}{1-\lambda}}\left(\frac{s}{n+\delta}\right)^{\frac{\lambda}{1-\lambda}}$$

$$\therefore y^* = A^{\frac{1}{1-\lambda}}\left(\frac{s}{n+\delta}\right)^{\frac{\lambda}{1-\lambda}}. \qquad (7)$$

Consumption per worker is $c_t = (1-s)y_t$ in any period t. So, the steady state equilibrium value of consumption per worker is:

$$c^* = (1-s)y^* = (1-s)A^{\frac{1}{1-\lambda}}\left(\frac{s}{n+\delta}\right)^{\frac{\lambda}{1-\lambda}}. \qquad (8)$$

The real rental rate for capital, r_t, and the real wage rate for labor, w_t, are given by the following two profit maximizing conditions of the competitive firms in this model:

$$r_t = \text{Marginal product of capital} = \frac{dY_t}{dK_t} = \lambda AK_t^{\lambda-1}N_t^{1-\lambda} = \lambda A\left(\frac{K_t}{N_t}\right)^{\lambda-1} = \lambda Ak_t^{\lambda-1}$$

and

$$w_t = \text{Marginal product of labor}$$
$$= \frac{dY_t}{dN_t} = (1-\lambda)AK_t^{\lambda}N_t^{-\lambda} = (1-\lambda)A\left(\frac{K_t}{N_t}\right)^{\lambda} = (1-\lambda)Ak_t^{\lambda}.$$

So, the steady state equilibrium values of r_t and w_t are:

$$r^* = \lambda A\left(k^*\right)^{\lambda-1} = \lambda A\left[A^{\frac{1}{1-\lambda}}\left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\lambda}}\right]^{\lambda-1} = \lambda AA^{-1}\left(\frac{s}{n+\delta}\right)^{-1} = \lambda\left(\frac{s}{n+\delta}\right)^{-1}$$

and

$$w^* = (1-\lambda)A\left(k^*\right)^{\lambda}$$

$$= (1-\lambda)A\left[A^{\frac{1}{1-\lambda}}\left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\lambda}}\right]^{\lambda} = (1-\lambda)AA^{\frac{\lambda}{1-\lambda}}\left(\frac{s}{n+\delta}\right)^{\frac{\lambda}{1-\lambda}} = (1-\lambda)A^{\frac{1}{1-\lambda}}\left(\frac{s}{n+\delta}\right)^{\frac{\lambda}{1-\lambda}}.$$

Figure B1(b) illustrates the steady state equilibrium levels of capital per worker, output per worker and consumption per worker.

(c) Find the growth rates of aggregate output per labor, aggregate capital per labor, aggregate output and aggregate capital on the balanced growth path. [2 marks]

On the balanced growth path (at the steady state equilibrium) the level of aggregate capital per worker is: $k^* = A^{\frac{1}{1-\lambda}}\left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\lambda}}$, which is a constant. So, the growth rate of k^* is zero.

Similarly, on the balanced growth path (at the steady state equilibrium) the level of aggregate output per worker is: $y^* = A^{\frac{1}{1-\lambda}}\left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\lambda}}$, which is a constant. So, the growth rate of y^* is zero.

Let the approximate growth rates of Y_t, y_t, K_t, k_t, and N_t from period $t-1$ to t be denoted by g_t^Y, g_t^y, g_t^K, g_t^k, and g_t^N. Using the definition of y_t we can express the aggregate output as,

$$Y_t = y_t N_t \qquad (9)$$

Taking logs on both sides of (9),

$$\ln Y_t = \ln y_t + \ln N_t \qquad (10)$$

So, we can also write,

$$\ln Y_{t-1} = \ln y_{t-1} + \ln N_{t-1} \qquad (11)$$

Subtracting (11) from (10) we get,

$$\ln Y_t - \ln Y_{t-1} = \ln y_t - \ln y_{t-1} + + \ln N_t - \ln N_{t-1}$$
$$\Rightarrow g_t^Y = g_t^y + g_t^N$$
$$\Rightarrow g_t^Y = g_t^y + n$$

Since on the balanced growth path g_t^y is zero, the growth rate of aggregate output, g_t^Y on the balanced growth path should be equal to the growth rate of labor, *n*.

Using the definition of k_t we can express the aggregate output as,

$$K_t = k_t N_t \qquad (12)$$

Taking logs on both sides of (12),

$$\ln K_t = \ln k_t + \ln N_t \qquad (12)$$

So, we can also write,

$$\ln K_{t-1} = \ln k_{t-1} + \ln N_{t-1} \qquad (13)$$

Subtracting (13) from (12) we get,

$$\ln K_t - \ln K_{t-1} = \ln k_t - \ln k_{t-1} + + \ln N_t - \ln N_{t-1}$$
$$\Rightarrow g_t^K = g_t^k + g_t^N$$
$$\Rightarrow g_t^K = g_t^k + n$$

Since on the balanced growth path g_t^k is zero, the growth rate of aggregate capital, g_t^K on the balanced growth path should be equal to the growth rate of labor, *n*.

(d) Find the elasticity of the steady state equilibrium value of aggregate output per worker with respect to the saving rate. What is the economic interpretation of this elasticity if the value of $\lambda = \frac{1}{4}$. [4 marks]

The steady state equilibrium value of aggregate output per worker, y^*, is:

$$y^* = A^{\frac{1}{1-\lambda}}\left(\frac{s}{n+\delta}\right)^{\frac{\lambda}{1-\lambda}}. \qquad (7)$$

Taking logs on both sides of (7) gives:

$$\ln y^* = \frac{1}{1-\lambda}\ln A + \frac{\lambda}{1-\lambda}\left[\ln s - \ln(n+\delta)\right]. \qquad (14)$$

The elasticity of y^* with respect to the saving rate, $s \equiv \frac{d\ln y^*}{d\ln s} = \frac{\lambda}{1-\lambda} = \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{3}$.

This means that according to this Solow Model, a 10 percent increase in the saving rate (investment rate) will imply an increase in people's average incomes of around 3.33% in the long run.

(e) Find the golden rule savings rate and the golden rule level of aggregate capital per worker. Solve for the steady state equilibrium values of aggregate output per worker and consumption per worker assuming that the exogenous saving rate is equal to the golden rule savings rate. Illustrate the golden rule levels of aggregate capital per worker, aggregate output per worker and consumption per worker in a diagram. [8 marks]

The steady state equilibrium value of consumption per worker is:

$$c^* = (1-s)A^{\frac{1}{1-\lambda}}\left(\frac{s}{n+\delta}\right)^{\frac{\lambda}{1-\lambda}}. \qquad (8)$$

The golden rule saving rate, s^{**}, is the steady-state consumption maximizing saving rate. So, to find s^{**} we have to maximize c^* with respect to s using (8). We are allowed to take logs before maximizing. Taking logs on both sides of (8) gives:

$$\ln c^* = \ln(1-s) + \frac{1}{1-\lambda}\ln A + \frac{\lambda}{1-\lambda}\ln s - \frac{\lambda}{1-\lambda}\ln(n+\delta).$$

So, the first-order condition of this maximization problem:

$$
\begin{aligned}
&\frac{\partial \ln c^*}{\partial s} = -\frac{1}{1-s} + \frac{\lambda}{1-\lambda}\frac{1}{s} = 0 \\
&\Rightarrow \frac{\lambda}{1-\lambda}\frac{1}{s} = \frac{1}{1-s} \\
&\Rightarrow \lambda(1-s) = s(1-\lambda) \\
&\Rightarrow \lambda - \lambda s = s - \lambda s \\
&\therefore s^{**} = \lambda.
\end{aligned}
$$

So, the golden rule saving rate is λ.

To find the golden rule level of capital per worker, k^{**}, we have to substitute s with λ in (6):

$$k^{**} = A^{\frac{1}{1-\lambda}}\left(\frac{\lambda}{n+\delta}\right)^{\frac{1}{1-\lambda}}$$

To find the golden rule level of output per worker, y^{**}, we have to substitute s with λ in (7):

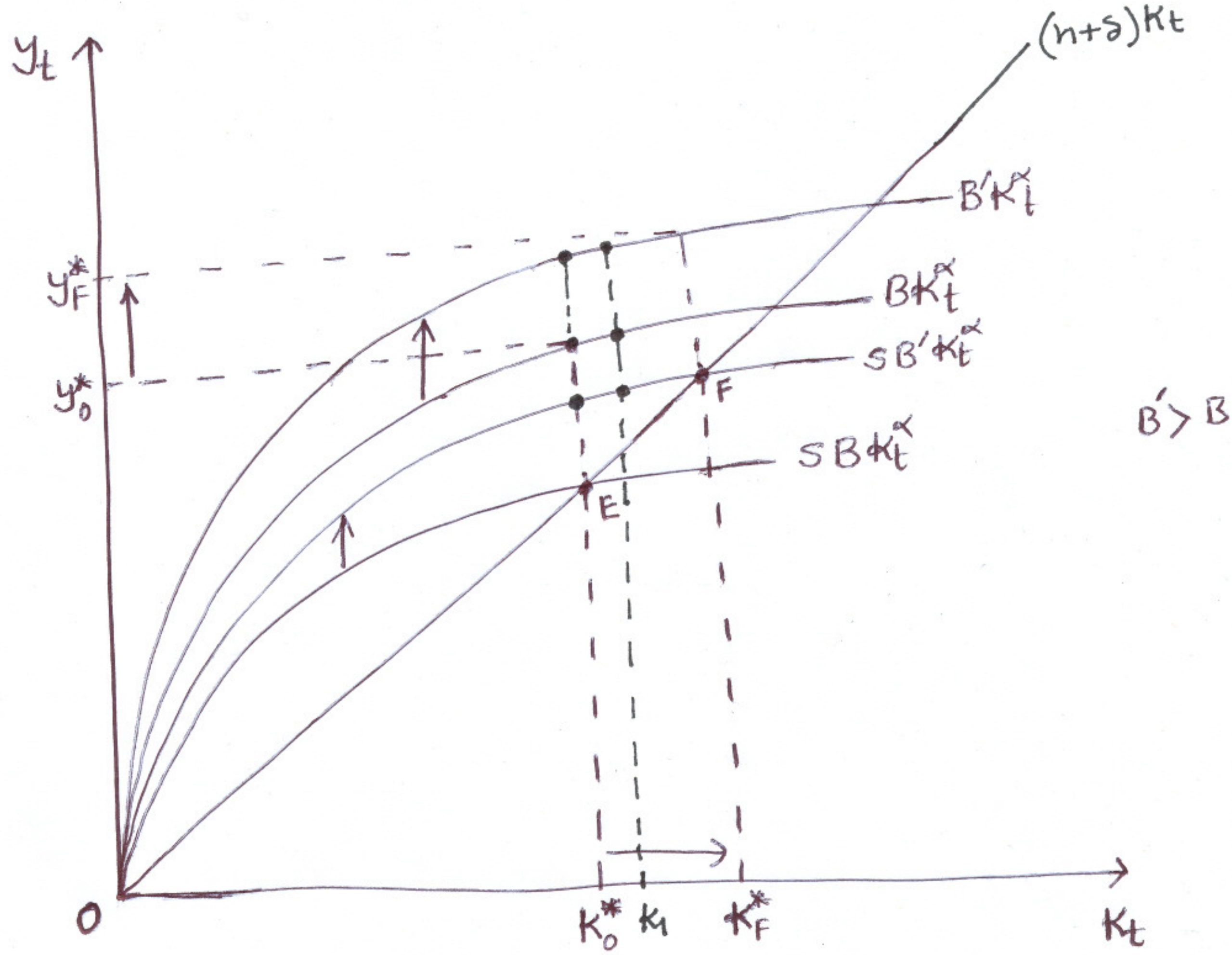
$$y^{**} = A^{\frac{1}{1-\lambda}}\left(\frac{\lambda}{n+\delta}\right)^{\frac{\lambda}{1-\lambda}}.$$

To find the golden rule level of consumption per worker, c^{**}, we have to substitute s with λ in (8):

$$c^{**} = (1-s)A^{\frac{1}{1-\lambda}}\left(\frac{\lambda}{n+\delta}\right)^{\frac{\lambda}{1-\lambda}}.$$

Figure B1(e) illustrates the golden rule levels of capital per worker, output per worker and consumption per worker.

FIGURE A2: THE EFFECTS OF AN INCREASE IN THE LEVEL OF TOTAL FACTOR PRODUCTIVITY (B)



INITAL STEADY STATE VALUES

$$K_0^* = B^{\frac{1}{1-\alpha}} \left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}}$$

$$y_0^* = B^{\frac{1}{1-\alpha}} \left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

NEW HIGHER STEADY STATE VALUES

$$K_F^* = (B')^{\frac{1}{1-\alpha}} \left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}}$$

$$y_F^* = (B')^{\frac{1}{1-\alpha}} \left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

FIGURE B1(a): THE TRANSITION EQUATION

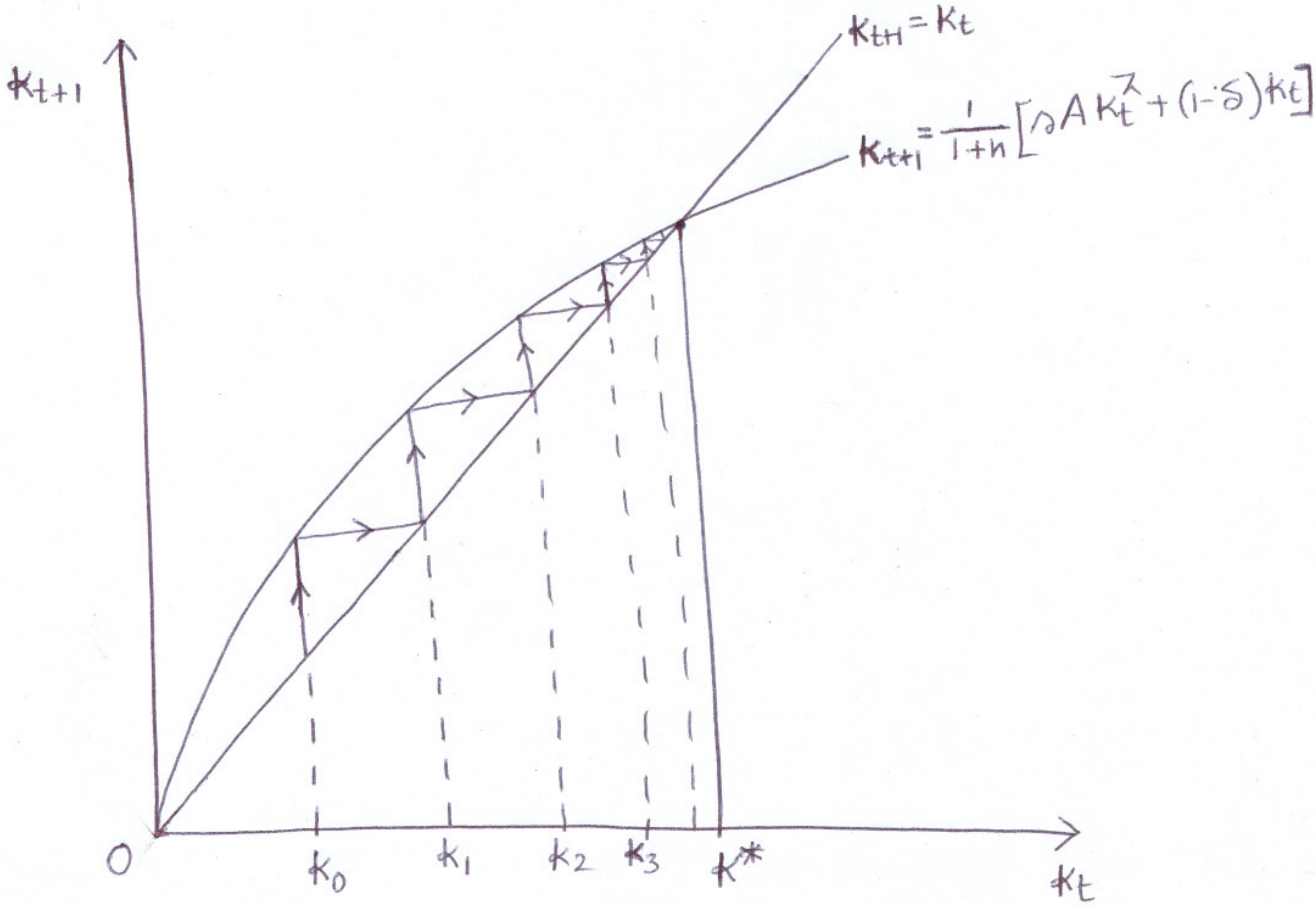


FIGURE B1(b): THE STEADY-STATE EQUILIBRIUM

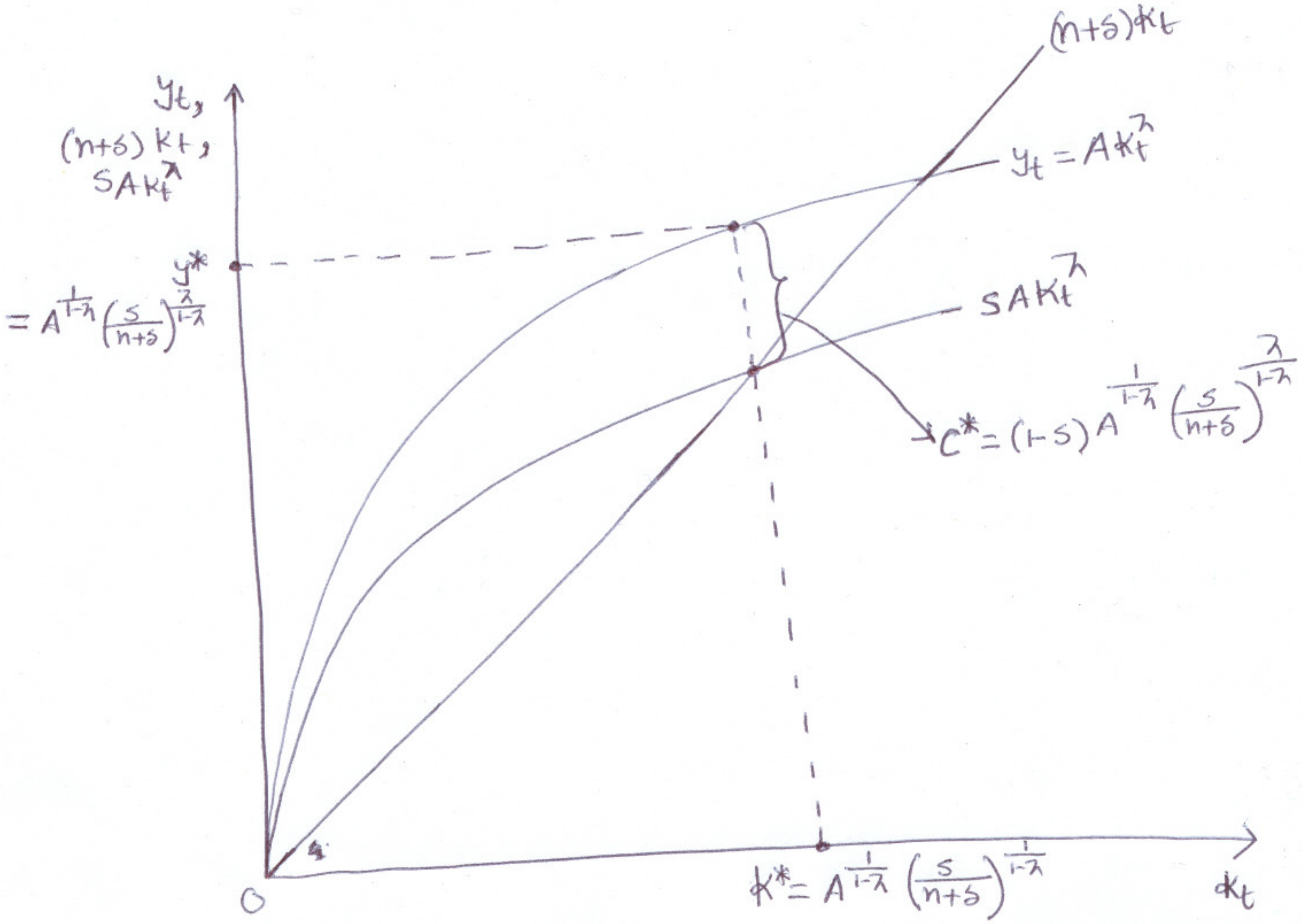


FIGURE B1(e): THE STEADY STATE EQUILIBRIUM WITH THE GOLDREN RULE SAVING RATE