6. CONSUMPTION AND TAX SMOOTHING

It is better to have a permanent income than to be fascinating.
– Oscar Wilde (1854-1900)

The hardest thing in the world to understand is income tax.
– Albert Einstein (1879–1955)

This section focuses on theories describing aggregate consumption spending and how it is related to income and interest rates. Consumption is an important component of aggregate demand, and hence central to our understanding of business cycles. As we'll see, one of the leading theories of how households consume also can be thought of as a theory of how governments set taxes.

(a) Budget Constraints and Present-Value Accounting

We'll begin with some properties of dynamic budget constraints and the notion of present-value budget balance, since it is relevant to lifetime budgeting. Consider the sequence budget constraint:

$$a_{t+1} = (1+r)a_t + y_t - c_t = (1+r)a_t + s_t \tag{1}$$

where there is no uncertainty, r is constant, and a is wealth, y is income, c is consumption, and s is saving. Note that y_t excludes interest income, which is given by ra_t. Sometimes we write r_t for the interest rate applying from period t to $t+1$, although one might want to alter that notation to r_{t+1}, depending on the nature of the specific financial investment.

To avoid confusion, let me mention that there are other possible ways to write down the timing in the budget constraint. A second one is

$$a_{t+1} = (1+r)(a_t + y_t - c_t)$$

and a third one is:

$$a_{t+1} = (1+r)a_t + y_{t+1} - c_{t+1}.$$

Technically, which one you use depends on the time at which the variables are measured, but nothing in our economic findings depends on which one we use, as long as we use it consistently.

Solving this difference equation gives the corresponding present-value budget constraint. Solving backwards gives:

$$a_t = \sum_{i=0}^{\infty}(1+r)^i s_{t-1-i} \tag{2}$$

provided that $lim_{i\to\infty}(1+r)^i a_{t-i} = 0$, which we shall assume. Starting life with zero wealth is sufficient for this limit to be zero.

An alternative is to solve the difference equation the other way round. This gives,

$$a_t = -\sum_{j=0}^{\infty}(1+r)^{-j-1}s_{t+j} \tag{3}$$

provided that $lim_{j\to\infty}(1+r)^{-j}a_{t+j} = 0$, which we shall assume. This transversality condition rules out unlimited lending or borrowing (bubbles) and hence dying in debt.

Equation (2) simply says that current wealth arises from past saving. Equation (3) says that current wealth can be used to finance future dissaving. It is feasible (satisfies the budget constraint) to consume more than income (*i.e.* $c_{t+j} > y_{t+j}$) in some future period if $a_t > 0$. In that case $s_{t+j} < 0$ so $-s_{t+j} > 0$ which is consistent with $a_t > 0$ in equation (3). So equation (3) is the present-value constraint, conditional on current assets.

Combining the two results gives:

$$\sum_{i=-\infty}^{\infty}(1+r)^{-i-1}s_{t+i} = 0, \tag{4}$$

which is the lifetime present-value budget constraint.

The same accounting applies to a government. Suppose that b_t is government debt or bonds outstanding, and s_t is the *primary* surplus, which equals $t_t - g_t$, where t is revenue and g is spending. The standard budget constraint or financing identity is:

$$\Delta b_t = rb_{t-1} + g_{t-1} - t_{t-1}$$

so that spending and interest payments in excess of revenue must be financed by issuing new bonds. This becomes

$$b_{t+1} = (1+r)b_{t-1} - s_{t-1}$$

since a surplus reduces liabilities. Note the difference from equation (1). For this agent, the government, we are measuring liabilities, rather than assets *i.e.* $b_t = -a_t$ for the government (although b_t is an asset, such as a bond, for the private sector).

The analogues to equations (2)-(3) now describe how past deficits lead to positive debt and how inherited debt constrains future fiscal policy. The transversality condition rules out debt bubbles, in which the government meets the interest payments on its maturing debt by issuing more debt. However, with an infinite horizon the debt does not need to be paid off – it can grow at rate less than r. See if you can show that the transversality condition rules out a permanent, primary deficit *and* that the government can run a permanent deficit *inclusive* of interest payments *e.g.* $s = 0$, with outstanding debt.

What would an apparent failure to satisfy the transversality constraint mean? One possibility is that the economy is dynamically inefficient. A second possibility is that one of the series has been mismeasured, since accounting for government assets and liabilities is not easy. A third possibility is that a rejection shows that the current pattern of fiscal policy cannot be sustained.

Finally, exactly the same accounting applies to a country, and now links the stock of foreign debt to the trade balance.

(b) Euler Equation Evidence

In studying consumption so far all we have done is write down the lifetime budget constraint. To learn which consumption plan (of the many that satisfy the constraint) is chosen, we shall next look at an optimization problem. We shall suppose that agents are not constrained in labour or credit markets (*i.e.* there are no liquidity constraints). Suppose that an agent chooses $\{c_t\}$ to maximize the following functional:

$$E_0 \sum_{t=0}^{T} \beta^t u(c_t)$$

It is called a functional because it is a function (specifically, a discounted sum) of utility functions at each time t. u is the period or instantaneous utility function; this functional is additively separable in the u's. β is the discount factor, where $\beta = 1/(1+\theta)$, and θ is the discount rate. A positive θ reflects some impatience or time preference. The function u sums up the degree of substitution between consumptions (of some composite good) in different periods and also characterizes (through its concavity) the degree of risk aversion within a period (more on this below).

In several parts of the course we've focused on labour supply and consumption *given* prices and now we do the same sort of thing *i.e.* we focus on the household's plans rather than seek a complete general equilibrium. The results depend on the objectives and constraints, and by working them out we learn about those objectives and constraints indirectly. We imagine the consumer as maximizing this functional subject to the budget constraint:

$$a_{t+1} = (1+r)a_t + y_t - c_t,$$

where y_t denotes labour income.

To avoid confusion, let us use the term *life-cycle/permanent-income hypothesis* (lcpih) to refer to the idea that observed aggregate consumption can be described as the outcome of this kind of optimization. I should warn you that this may not be standard. Some writers use the term to refer to a particular linear example (from quadratic utility) which we shall see in a moment.

Just as in the two-period model (which is a special case) one of the first-order necessary conditions for a maximum in this optimization problem is the Euler equation:

$$u'(c_t) = E_t \beta (1 + r_{t+1}) u'(c_{t+1})$$

No matter what assets are available for transferring wealth between time t and time $t+1$, in equilibrium their rates of return must satisfy this equation. Or, for the household r is exogenous and choices of endogenous c must satisfy this equation. So depending on our emphasis we can view these equations (with rates of return on various assets) either as a theory of consumption and saving decisions or a theory of asset returns, as we saw in section 2. In fact they are both, but it is convenient to take one perspective at a time.

Let us consider several examples of this Euler equation.

Example 1

$$u(c_t) = ac_t - bc_t^2$$

with $a > 0$, $u' > 0$ and $u'' < 0$; and suppose that $\theta = r$, a constant. Then the Euler equation is:

$$c_{t+1} = c_t + \epsilon_{t+1} \qquad E_t\epsilon_{t+1} = 0$$

This is a very special example – with a constant interest rate and quadratic preferences – which says that, given c_t, no other information available at time t should help predict the value of c_{t+1}. The idea is that agents base consumption on lifetime or permanent income. They do this now, and hence c_t summarizes all information available now on future income prospects. If c_{t+1} differs from c_t it must be due to new information not available at time t.

One of the major developments in dynamic, stochastic macroeconomics (as mentioned in section 5) has been the insight that complete solutions to models need not be found in order to test them. The first-order condition given here can be used to test the optimization model without a complete solution for c_t in terms of variables such as expected future income and interest rates (let alone exogenous shocks). Since it relates one endogenous variable to another, instrumental variables methods are used to estimate it.

Hall (1978) noticed that the Euler equation implies that discounted marginal utility is a martingale: future changes in marginal utility are not predictable from anything in the agent's current information set. He tested this by methods made familiar by applied research in macro based on rational expectations and in finance on efficient markets. If we knew what marginal utilities were, call them $mu_t = u'(c_t)$, and if the rate of interest were constant, we could estimate regressions of the form

$$mu_{t+1} = \gamma_1 mu_t + \gamma_2 z_t + \epsilon_{t+1}$$

where z_t is anything in the agent's date-t information set and ϵ_{t+1} is a forecast error with conditional mean zero: $\mathrm{E}_t\epsilon_{t+1} = 0$. The first parameter, γ_1, equals $\beta(1+r)$ and is expected to be close to one, especially for short time intervals. The theory implies $\gamma_2 = 0$, which can be used as the basis of a statistical test. Hall made this operational by assuming that marginal utility is linear in consumption, (as would be the case with quadratic utility) so that a similar equation can be estimated with c_t replacing mu_t.

Hall considered three candidates for z. The first, consumption lagged more than once, had little predictive power for changes in consumption. The second, stock prices, provided moderately strong evidence against the random walk model. Higher stock prices tend to be associated with larger changes in consumption. This turns out to be a robust result, and may suggest either that the first-order condition is an imperfect description of aggregate data or that the assumption of constant interest rates is inadequate. The third choice of z was lagged income. Hall found that lagged changes in income lead to an improvement in predictions of changes in consumption.

Let us consider this role for current income from the perspective of an alternative model suggested by Hall. Suppose there are two groups of consumers in the economy. The first (group a) consume according to the lcpih (quadratic version), and receive a fraction $1-\mu$ of aggregate income. For them, $\mathrm{E}_t c^a_{t+1} = c^a_t$ is a good approximation. The second group simply consumes all their income, $c^b_t = \mu y_t$. If income is a first-order autoregression, *i.e.*

$$y_{t+1} = \rho y_t + v_{t+1},$$

then we can derive the behavior of aggregate consumption, $c_t = c^a_t + c^b_t$, as follows:

$$\begin{aligned}
\mathrm{E}_t c_{t+1} &= \mathrm{E}_t c^a_{t+1} + \mathrm{E}_t c^b_{t+1} \\
&= c^a_t + \mu \mathrm{E}_t y_{t+1} \\
&= (c_t - c^b_t) + \mu\rho y_t \\
&= c_t + \mu(\rho - 1) y_t.
\end{aligned}$$

The intuition is that changes in the second group's consumption are predictable if changes in their income are. This leads to predictability in aggregate consumption as long as $\mu > 0$ and $\rho\mu \neq 1$. For Canada the evidence suggests that $\hat{\mu} = .20$ or that 20% of consumption is by consumers in the second category. I have simplified this story in a way which leads to a negative coefficient on y_t, which is not found empirically – but that would change with a more realistic time series model for y_t, involving some growth or trend.

This provides an interpretation of income in the regression: some fraction of the population is *liquidity constrained*. The idea is to conduct empirical tests of Euler equations using variables z_t which should be related to liquidity constraints, and to see whether they can predict the innovation or error in the Euler equation, which should be unpredictable. Unfortunately, most of these tests do not specify an alternative hypothesis. The Euler equation (as we have seen) could fail for several reasons. The precise nature of the constraint (it could be that the borrowing rate exceeds the lending rate, or that there is a quantity constraint on borrowing) affects the implications. Moreover μ should probably *not* be viewed as an estimate of the fraction of the population who are constrained (since there is not a complete model with two agents and trade here – an economy with μ constrained agents and $1-\mu$ unconstrained is different from μ of a constrained economy and $1-\mu$ of an unconstrained one), but simply as a measure of the failure of this version of the lcpih.

Example 2

$$u(c_t) = \ln c_t.$$

The Euler equation is

$$\frac{1}{c_t} = E_t\Big[\frac{\beta(1+r_{t+1})}{c_{t+1}}\Big].$$

So far we have examined implications of the optimising model for the properties of consumption. We mentioned that time variation in interest rates would complicate tests of the lcpih. But it also is possible to test the Euler equation and the corresponding theory by examining the relation between consumption and interest rates, as section 2 described for two-period models. One of the most interesting aspects of macroeconomics concerns the interaction of financial markets and real decisions, and the relation between consumption and asset returns is perhaps the most fundamental example of this interaction.

As we have noted before, some care with the timing notation is required. The measure r is the interest rate from time t to time $t+1$. For certain assets (*e.g.* equities) that return is uncertain at time t and the uncertainty is not resolved until time $t+1$; hence r_{t+1} is the appropriate notation. For other assets (*e.g.* T-bills) the rate of return is known at time t so that writing r_t is appropriate.

Consider the log preferences of example 2 – at least some empirical work suggests that this may not be far wrong. For simplicity I shall ignore uncertainty for a moment:

$$\begin{aligned} c_t^{-1} &= \beta(1+r_t)c_{t+1}^{-1} \\ \frac{c_{t+1}}{c_t} &= \beta(1+r_{t+1}) \\ \ln(\frac{c_{t+1}}{c_t}) &= \ln\beta + r_{t+1} \end{aligned}$$

I have used the approximation $\ln(1+x) \approx x$ for small x. This equation says that the growth rate ($\Delta \ln c_t$) of consumption should be equal to a constant (a small negative number if $\beta \in (0,1)$) plus the interest rate. In periods in which the interest rate is high consumption should be growing rapidly. The idea is simply that if there is a large return to saving (deferred consumption) then consumption will be postponed. This specific property depends on the logarithmic preferences, which mean that there is a great deal of intertemporal substitution in consumption. One can imagine alternative (*e.g.* Leontief) preferences in which c_{t+1} can in no way substitute for c_t – in that case the growth rate of consumption will be insensitive to market opportunities or price changes as characterized by r_{t+1}.

In this example, the one-for-one response arises from the log function. More generally, the degree of response can tell us something about preferences. This equation can be examined readily in time series data for different periods and frequencies. Often there is little evidence of intertemporal substitution in consumption.

Example 3

$$u(c_t) = c_t^{1-\alpha}/(1-\alpha).$$

This period utility function often is used in studies of consumption and of asset prices. The coefficient of relative risk aversion is $|u''(c)c/u'(c)| = \alpha$. The parameter α is positive, for concavity. This class of instantaneous utility functions are referred to as constant relative risk aversion or CRRA utility functions. In fact, example 3 generalizes example 2; if $\alpha = 1$ then $u(c) = \ln(c)$. The Euler equation is:

$$c_t^{-\alpha} = E_t[\beta(1 + r_{t+1})c_{t+1}^{-\alpha}].$$

If I again ignore uncertainty,

$$\Delta \ln c_t = \frac{\ln \beta}{\alpha} + \frac{r_{t+1}}{\alpha}$$

so that the elasticity of intertemporal substitution ($\Delta \ln$ quantity divided by $\Delta \ln$ price) is the inverse of the coefficient of relative risk aversion. Given these preferences, if consumption responds very little to changes in the interest rate that implies high risk aversion or low intertemporal substitution.

Wirjanto (1991) examined regressions such as

$$\Delta \ln c_t = \gamma_0 + \gamma_1 \Delta \ln y_{t-1} + \gamma_2 r_t,$$

in quarterly Canadian data from 1953 to 1986; r is the 3-month, ex post real Treasury-Bill rate, c_t is real per capita consumption on non-durable goods and services, and y_t is real per capita disposable income. He finds that $\hat{\gamma}_1$ is roughly 0.2 (which rejects the random walk model) and that $\hat{\gamma}_2$ is roughly .08 so that α is 12.5. That implies a lot of risk aversion or a little intertemporal substitution in consumption. What do the indifference curves look like, for consumption in adjacent time periods? This result means that fiscal and monetary policies which influence r (after-tax) may not have very much effect on saving.

Notice also that the theory links the variance of consumption growth to the variance of interest rates. Informally,

$$\text{var}(\Delta \ln c_t) \simeq (\frac{1}{\alpha})^2 \text{var}(r_t).$$

We know that interest rates are more variable than consumption growth rates, which implies that α also must be large. Sometimes consumption is said to be too smooth relative to asset prices. That simply means that the value of α, for example, which can match the consumption variability with that of interest rates is implausibly large.

Example 4

$$u(c_t) = -exp(-\alpha c_t)/\alpha,$$

or CARA utility. The Euler equation is:

$$exp(-\alpha c_t) = E_t[\beta(1 + r_{t+1})exp(-\alpha c_{t+1})].$$

With this form of utility function we can solve for the closed-form consumption function if we assume that income shocks are normally distributed.

Studying an Euler equation is easy, but if tests reject it then we don't necessarily know how to reformulate the model. To learn more, we can study the consumption function. In some special cases we can solve for this function analytically.

(c) Consumption Functions

With quadratic utility and a constant interest rate the Euler equation is linear, so it's easy to combine it with the budget constraint. That constraint is:

$$a_t = -\sum_{j=0}^{\infty} E_t(1+r)^{-j-1} s_{t+j}$$

or, rewriting,

$$a_t + \sum_{j=0}^{\infty}(1+r)^{-j-1} E_t y_{t+j} = \sum_{j=0}^{\infty}(1+r)^{-j-1} E_t c_{t+j}.$$

When $E_t c_{t+1} = c_t$ it's also true that $E_t c_{t+j} = c_t$, using the law of iterated expectations. Thus,

$$c_t = r[a_t + \sum_{j=0}^{\infty}(1+r)^{-j-1} E_t y_{t+j}],$$

which is the consumption function.

Under the lcpih lifetime income matters and not its composition. The individual consumes the annuity value of expected lifetime labour income and current wealth. To see what I mean by annuity value, suppose that there is no labour income but only financial wealth a_t, as if the agent has retired but will live forever, like a trust fund. How much should be consumed? Here

$$c_t = r a_t$$

while the budget constraint is

$$a_{t+1} = a_t(1+r) - c_t$$

Combining these gives:

$$a_{t+1} = a_t(1+r) - r a_t = a_t$$

This prudent (and infinitely-lived!) consumer spends the net interest income but maintains wealth constant. In this special example, the marginal propensity to consume out of wealth is the constant interest rate, r.

To make the theory testable, we need to model the expectations of future labour income, using the forecasting tools we developed in section 3. Suppose, for example, that

$$y_t = \rho y_{t-1} + v_t$$

$E_{t-1} v_t = 0$ and $|\rho| < 1$ Recall that in this case:

$$E_t y_{t+j} = \rho^j y_t.$$

The consumption function now is:

$$
\begin{aligned}
c_t &= r[a_t + \frac{y_t}{1+r} + \frac{\rho y_t}{(1+r)^2} + ...] \\
&= ra_t + \frac{r}{1+r-\rho} y_t.
\end{aligned}
$$

Exercise: Write this problem as a dynamic programme, and solve for the consumption function using the guess-and-verify method. (This is exactly like using Method A instead of Method B in section 3).

We still face an obstacle in testing this consumption function: assets a_t are very difficult to measure accurately. But we can avoid this obstacle with a trick. Lag the consumption function and multiply it by $1+r$:

$$
(1+r)c_{t-1} = r(1+r)a_{t-1} + (1+r)\frac{r}{1+r-\rho} y_{t-1}.
$$

The trick is that the original consumption function included a_t while this version includes $(1+r)a_{t-1}$. But the budget constraint is:

$$
a_t - (1+r)a_{t-1} = y_{t-1} - c_{t-1}.
$$

Thus if we subtract the version that is lagged and multiplied by $(1+r)$ from the original, then only the flows of income and consumption will remain. The result of this subtraction is:

$$
c_t = c_{t-1} + \frac{r}{1+r-\rho}(y_t - \rho y_{t-1}).
$$

When you recall that $y_t - \rho y_{t-1} = v_t$, our result looks suspiciously like our original Euler equation, in which the change in consumption was unpredictable. But we now have a deeper understanding of revisions in the consumption plan:

$$
\begin{aligned}
\epsilon_t &= \frac{r}{1+r-\rho} v_t \\
&= r \sum_{j=0}^{\infty} (1+r)^{-j-1}(E_t y_{t+j} - E_{t-1} y_{t+j}).
\end{aligned}
$$

It may take you a few minutes to confirm this last result. The idea is that a shock to current income, v_t, leads to revisions in forecasts of future income because the income series is persistent. So to see how much to change consumption, you first have to recalculate the present value of income. Then the change in the annuity value is the change in consumption.

In the consumption function we've found, $\rho = 1$ makes all income changes permanent and $\rho = 0$ makes them completely temporary. Thus actual consumption changes depend in a specific way on contemporaneous and past labour income. This shouldn't surprise you. With these preferences the agent tries to smooth consumption and to set it based on expected lifetime income. The reason why c_t differs from c_{t-1} is that new information on lifetime income arrives between $t-1$ and t. That information is contained in y_t, which allows the agent to update forecasts for future incomes as well. Of course, that updating is not predictable at time $t-1$.

The model can be tested by running a set of linear regressions:

$$\begin{aligned} c_t &= c_{t-1} + (\frac{r}{1+r-\rho})(y_t - \rho y_{t-1}) + e_t \\ y_t &= \rho y_{t-1} + v_t \end{aligned}$$

where e_t reflects various possible misspecifications. Notice that the theory restricts this simple regression system quite severely. There are three regressors in the first equation (the consumption function) and one in the second (the income forecasting equation) but only two parameters to be estimated, namely ρ and r. The system is overidentified and can thus be tested by seeing whether two values for the parameters can explain all four reduced-form coefficients.

What is the evidence? An influential study by Flavin (1991) found that the coefficient on y_{t-1} was too large to be consistent with the theory, so consumption is more closely tied to past labour income than the theory predicts. She referred to this finding as *excess sensitivity* of consumption.

A complementary finding by Campbell and Deaton (1989) was that consumption also displays *excess smoothness*. Recall our result that

$$\Delta c_t = (\frac{r}{1+r-\rho})v_t$$

Then taking the variances of each side:

$$\mathrm{var}(\Delta c_t) = \left(\frac{r}{1+r-\rho}\right)^2 \mathrm{var}(v_t)$$

In most aggregate data (with estimates of r and ρ) the actual value of $\mathrm{var}(\Delta c_t)$ is smaller than the right-hand side of this equation. In that sense, consumption is too smooth to be consistent with the lcpih.

To take an example, suppose that $\rho = 1$. In that case labour income is said to have a unit root or to be integrated. The idea is that the forecasting equation for income is just a difference equation with an error term added, and the root of that equation is one. If that is so then, solving backwards:

$$y_t = v_t + \rho y_{t-1} = v_t + v_{t-1} + v_{t-2} + \dots$$

so that shocks (the v's) to labour income are permanent. In practice estimates of ρ are near one, so that the coefficient in the regression of Δc_t on v_t should be large (also one).

That makes sense: if innovations to one's labour income are permanent then one should adjust one's consumption by a large amount. However, the actual regression suggests that consumption is not sufficiently responsive to these innovations in labour income.

In summary, the empirical evidence suggests that the quadratic version of the lcpih is inconsistent with the evidence because consumption changes are overly sensitive to expected income (*i.e.* can be predicted by y_{t-1}) whereas other evidence suggests that consumption changes are not sensitive enough to unanticipated income. One possible explanation is that there are liquidity constraints; another is that more complex (nonlinear) versions of the model, perhaps with variation in interest rates, are needed.

In the case of quadratic preferences we have been able to find both the Euler equation and the consumption function. But remember that example 1, with quadratic utility and a constant $r = \theta$, is not very general. In particular:

▹ In the closed-form solution for c_t given above the current level of consumption depends only on expected future income and not on uncertainty about that income. This is the property of *certainty equivalence.* With other functional forms for u (in which $u''' \neq 0$) the optimal c_t also depends on the variability of the income stream. Typically, greater uncertainty (with the same expected value) about future income leads to reduced consumption currently and more precautionary saving.

▹ We have assumed that there are no taste shocks, that is, that the utility function itself is not subject to random movements. Hence, we regard data as generated by a budget line shocked against a constant indifference curve. Taste shocks may explain Christmas shopping and other seasonal effects but they are assumed not to explain business cycles.

▹ We have assumed that the utility functional is additively separable over time; thus utility at time t depends only on consumption at time t. Preferences inconsistent with this assumption would include those in which there is habit persistence, for example. This assumption of separability becomes increasingly tenuous as the observation interval (indexed by t) becomes small. Your happiness this afternoon may not depend on what you ate last year but may depend on this morning's breakfast.

▹ We have also assumed that there is one source of consumption, say a single good. But c_t should represent the flow of consumption services, whereas in empirical tests we must generally identify it with expenditures on consumption goods. If these goods have some durability then this match may be misleading. So it seems sensible to test the model with data which exclude spending on durables. Measurement error in consumption or income also will affect tests.

▹ We also have ignored leisure *i.e.* assumed that the period utility function is additively separable in goods. Suppose that $u = u(c_t, l_t)$ where l_t is leisure at time t. If consuming the consumption good(s) and consuming leisure are complementary activities then $\partial u(c_t, l_t)/\partial c_t$, the marginal utility in the first-order conditions above, will depend on leisure as well as consumption of the good. In that case, the model will hold predictions/restrictions for the joint behaviour of leisure and consumption over time. There can be a permanent income theory of leisure, for example.

▷ We have omitted variation in interest rates.

(d) Tax Smoothing

In section 2 of this course we studied the effects of a change in tax timing when there are lump sum taxes. One result was that under Ricardian equivalence there would be no reason for a government to choose any particular path for the budget deficit. Next, we'll consider this choice when the only taxes available are distorting income or commodity taxes. The result will be a set of predictions called the *tax smoothing* model. We'll continue to take the sequence of government spending as given.

Suppose that the deadweight loss of consumer surplus from taxation in a given period is a convex function, L, of total tax revenue t. This assumption means that tax take of \$100 in one year and \$100$(1+r)$ in the next is preferable to one (equivalent in present value terms) of \$200 in the first year. In these circumstances, smooth taxes will be best, just as uniform commodity taxes acros goods are sometimes best in public finance theory. Agents have access to perfect capital markets for transferring income through time; the idea is rather that

∘ the elasticity of labour supply doesn't vary over time, or

∘ costs of collection are convex (or agents can legally avoid taxes in a given year by transferring income across years).

Suppose, for simplicity though this is not necessary, that total output, y, though not welfare, is independent of the proportional tax rate t and constant over time. And all variables are real, for simplicity. Suppose that the stream of government expenditures $\{g_t : t = 0, 1, 2, ...\}$ is given. The government solves:

$$\min E_t \sum_{t=0}^{\infty} \beta^t L(t_t)$$

subject to:

$$b_{t+1} + t_t = (1 + r_t)b_t + g_t; \quad t_t = \tau_t y_t$$

where g is measured exclusive of interest payments. The Euler equation is:

$$L'_t = E_t \beta(1 + r_{t+1})L'_{t+1}$$

which is exactly like the permanent-income model of consumption, but with different labels on the variables.

For example, consider the case in which L is quadratic and $\beta \cong (1 + r)^{-1}$. In that case,

$$E_t \tau_{t+1} = \tau_t$$

or, $\tau_{t+1} = \tau_t + \epsilon_{t+1}$ and $E(\tau_t \cdot \epsilon_{t+1}) = 0$ so that there is tax smoothing. This *positive* theory predicts that deficits will be used to smooth tax rates over time.

This Euler equation can be tested statistically. Moreover, it can be combined with the government budget constraint just as was the case in finding the consumption function. In this case, we shall find a permanent-spending theory of tax rates.

One way to think about the implications of this theory is to introduce unexpected temporary or permanent shocks to g and trace out the behaviour of τ. Let us begin with a temporary shock. During a war, when g is temporarily large, the government should run a deficit and then pay it off with a series of small surpluses later. The numerical value of the tax rate will be determined by the government budget constraint, given the stream of expected future government purchases. Try an example to see how this works. You will see that the response to anticipated shocks may provide a stricter test of the theory.

Next, if there is a permanent increase in g the tax rate should jump up to reflect this new information. In these thought experiments the present value of government spending is used in the budget constraint to give the level of the constant (expected) tax rate. A government following this strategy would increase τ so as to raise t one-for-one in response to a permanent increase in g. But the response to a temporary increase in g would be smaller, as the necessary revenue is collected smoothly over time and the initial deficits are paid off. If there is an expected temporary increase in g then tax rates should rise by a small amount as soon as the future increase is expected, again to smooth τ. Thus this positive theory of taxes becomes a theory of deficits, if we think of government spending as being given.

This model fits the empirical facts that $\{t_t\}$ tends to be smoother than $\{g_t\}$ and that wars are often deficit-financed. But, one worries that the sequence $\{g_t\}$ should come from the same optimization problem. We generally do not model firms as deciding on investment decisions and then considering how to finance them, and one suspects that many spending decisions are influenced by current tax revenue and tax rates and also by deficits. Moreover, shocks which cause cycles in this model are unrelated to the convexity of L by assumption–that is, a shock which causes income/output to fall does not affect the relative elasticities of demand that underlie calculations of optimal tax rates. Also notice that optimal tax rates are uniform across periods so that deficits are countercyclical but not for conventional Keynesian reasons.

The tax smoothing model treats the authorities as solving a problem of commodity taxation only. When there is debt and capital in the economy the government's incentives may be more complex than we have assumed so far. For example, when the government has outstanding debt it may levy a lump-sum tax simply by defaulting on its debt. Likewise, if there is a positive capital stock a lump-sum tax may be levied on it. Of course, agents will be aware of the government's incentive to default on its debt or to tax capital, and that may discourage them from holding debt or investing in capital. In these circumstances the government is said to face a problem of *time consistency.* This simply means that if the government calculates time paths of taxes optimally by taking the behaviour of the private sector as given then it may have incentives to change its announced policy *e.g.* to promise zero taxes on capital to encourage investment and then subsequently to renege and tax capital since it is supplied inelastically.

However, the implication of this incentive for time inconsistency is not that we should observe reneging. In a rational expectations equilibrium agents will underinvest in capital if the government cannot bind itself not to apply capital levies in the future. The benevolent government has an incentive to tax capital now–since this will not affect output or allocations in the economy–and simultaneously persuade those investing in physical capital that it will not resort to this source of revenue in the future. Thus the optimal tax problem involves the search for means of *commitment* on the part of the government. If the government can find a method of binding itself in future actions then current agents may be made better off.

Sometimes, the government will have means to bind itself. For example, suppose that people hold less money than would be optimal because they believe the government will expand the money supply later, thus earning revenue from inflation since it is a large debtor in nominal terms. If the government issues only indexed debt, it will signal its removal of its own incentive to resort to the inflation tax and will thus encourage more money-holding, which may be efficient (of course, the maturity composition of government debt may depend also on capital market imperfections which it may try to exploit in order to finance its debt most cheaply). Establishing a central bank also may serve this purpose.

A similar problem in fiscal policy arises with regard to privatisation. Suppose that the government has two aims in privatising a public enterprise (such as Air Canada)–namely, raising revenue and promoting competition. If it wishes to raise a lot of revenue it will advertise and auction the enterprise in a regulatory environment that makes it a monopoly and therefore attractive to bidders seeking profit. If it seeks to encourage competition it will then renege and ensure deregulation. The government's inability to prohibit future deregulation may inhibit its ability to raise revenue currently.

Rational expectations equilibria (or sequential equilibria in games between the private and public sectors) do not involve predictable U-turns. Typically they involve a wide variety of equilibria, some of which can be sustained without commitment devices. More on these games of strategy as models of policy (and on the strategic role of policy coordination and signalling) is found in your friendly neighbourhood microeconomics course.

For further reading

Chapter 7 of David Romer's *Advanced Macroeconomics* (1996) surveys some topics in aggregate consumption. I warmly recommend section 6.2 of Blanchard and Fischer's *Lectures on Macroeconomics* (1989) and Angus Deaton's book *Understanding Consumption* (1992) to PhD students. Blanchard and Fischer discuss precautionary saving, neglected in these notes.

The original work on the random walk model was by Robert Hall in "Stochastic implications of the life cycle permanent income hypothesis," *Journal of Political Economy* (1978) 971-987, which M.A. students should read. The corresponding consumption function was studied in more technical work by Marjorie Flavin, "The adjustment of consumption to changing expectations about future income," *Journal of Political Economy* (1981) 974-1009. Excess smoothness was diagnosed by John Campbell and Angus Deaton in "Why is consumption so smooth?" *Review of Economic Studies* (1989) 357-373.

On liquidity constraints, see Fumio Hayashi's "Tests for liquidity constraints," 91-120 in Truman Bewley, ed. *Advances in Econometrics* (1987) fifth world congress, volume 2.

Canadian evidence on the permanent income hypothesis is provided by Tony Wirjanto in "Aggregate consumption behaviour and liquidity constraints," *Canadian Journal of Economics* (1995) 1135-1152 and in "Testing the permanent income hypothesis: the evidence from Canadian data," *Canadian Journal of Economics* (1991) 563-577.

The most interesting current research on consumption uses panel data, and leaves the representative-agent model behind. For an example see Angus Deaton and Christina Paxson's "Intertemporal choice and inequality," *Journal of Political Economy* (1994) 437-467.

For good introductions to tax smoothing see Sargent's essay "Interpreting the Reagan deficits," *Federal Reserve Bank of Minneapolis Quarterly Review* (Fall 1986) and Rao Aiyagari's "How should taxes be set?" *Federal Reserve Bank of Minneapolis Quarterly Review* (Winter 1989) V.V. Chari describes the commitment problem in "Time consistency and optimal policy design," *Federal Reserve Bank of Minneapolis Quarterly Review* (Fall 1988). For the underlying game theory, begin with chapter 15 of Varian's *Microeconomic Analysis* (1992).

Exercises

1. Show that adding white noise preference shocks to the quadratic utility model with $\beta(1+r)=1$ to give

$$u(c_t) = -(\alpha - \epsilon_t - c_t)^2$$

introduces a serially correlated (in fact, first-order moving average) error in the change in consumption.

2. Find the Euler equation in the two-period model with $r_{\text{borrow}} > r_{\text{lend}}$.

3. Suppose that preferences are quadratic and interest rates are constant, in the life cycle/permanent income model. Thus the Euler equation is given by

$$c_t = c_{t-1} + \epsilon_t; \qquad \mathrm{E}_{t-1}\epsilon_t = 0.$$

Suppose that ϵ_t arises from revisions in expectations, as follows:

$$\epsilon_t = \sum_{j=1}^{\infty} r(1+r)^{-j}\mathrm{E}_t y_{t+j} - r(1+r)^{-j}\mathrm{E}_{t-1} y_{t+j}.$$

Finally, suppose that income evolves according to

$$y_t = \rho y_{t-1} + v_t.$$

Suppose that $r = 0.01$ and that $\rho = 0.8$ in the economy. Suppose that an economist over-estimates the persistence in the income process, and thinks that $\rho = 0.9$. Can that mistake lead to findings of apparent (i) excess sensitivity (*i.e.* too large a response to anticipated or lagged income) (ii) excess smoothness (*i.e.* too small a response to unanticipated income)?

4. Suppose that labour income evolves as follows:

$$y_t = \eta + \rho y_{t-1} + u_t,$$

where $u_t \sim iid(0, \sigma^2)$. Suppose that consumption follows from the linear-quadratic version of the lcpih:

$$c_t = c_{t-1} + \epsilon_t,$$

$$\epsilon_t = \left(\frac{r}{1+r}\right)\sum_{j=0}^{\infty}(1+r)^{-j}(\mathrm{E}_t y_{t+j} - \mathrm{E}_{t-1} y_{t+j}).$$

Finally, suppose that $\rho = 1$. This is sometimes called a 'unit root'. The idea is that for countries like Canada income may grow on average (*i.e.* in this case $\Delta y_t = \eta + u_t$) rather than being stationary around some mean.

(a) Find ϵ_t, the innovation in consumption, as a function of u_t, the innovation in income.

(b) Find the variance of Δc_t and hence the ratio $var\Delta c_t/var\Delta y_t$. Does this model accord roughly with empirical evidence?

Answer

(a) $E_t y_{t+j} = y_t + j\eta = \eta + y_{t-1} + u_t + j\eta$, and $E_{t-1}y_{t+j} = y_{t-1} + \eta(j+1)$. So their difference is simply u_t. Thus

$$\epsilon_t = (\frac{r}{1+r})\sum_{j=0}^{\infty}(1+r)^{-j}u_t = u_t$$

(b) Thus $\Delta c_t = u_t$ so its variance is equal to the variance of income growth. This doesn't match up with the facts, where one tends to find that consumption is smoother than income *i.e.* a ratio less than one. This is simply the 'excess smoothness' puzzle.

5. This question examines tests of the linear-quadratic version of the lcpih model of aggregate consumption. Suppose that labour income evolves as follows:

$$\begin{aligned} y_0 &= 0 \\ y_t &= 0.9y_{t-1} + \epsilon_t;\ t = 1, 2, 3, \ldots, 50; \\ y_t &= 0.5y_{t-1} + \epsilon_t;\ t = 51, 52, 53, \ldots, 100. \end{aligned}$$

with $\epsilon_t \sim iin(0, 0.05)$. Notice that income becomes less persistent in the second half of the period. We have written the process in levels, but the same thing could be done with rates of change to allow for growth. Suppose that $r = 0.05$, a constant. Suppose that $c_0 = 0$ and

$$c_t = c_{t-1} + (\frac{r}{1+r-\rho})(y_t - \rho y_{t-1}). \qquad (*)$$

(a) On a computer, generate one replication of the labour income series.

(b) Using the ρ that applies for each time period, calculate the series for consumption.

(c) In this generated data, do a Hall-type test of the random walk model of consumption, by regressing Δc_t on a constant and y_{t-1}.

(d) Now assume that an econometrician does not realize that the labour income process has changed. Said econometrician observes the income series you have generated in part (a) and the consumption series you have generated (with the appropriate shift in ρ) in part (b). This person regresses y_t on y_{t-1} for the 100 observations to find $\hat{\rho}$. This person knows that $\rho = 0.05$ and then generates $\{\hat{c}_t\}$ using equation (*), with this r and $\hat{\rho}$. Plot the errors $\{c_t - \hat{c}_t\}$.

6. This question studies the properties of aggregate consumption from the perspective of the life-cycle/permanent-income hypothesis. Suppose that a representative agent maximizes

$$EU = E\sum_{t=0}^{\infty}(1+r)^{-t}(b - c_t)^2,$$

subject to the budget constraint

$$a_t = (1+r)(a_{t-1} + y_{t-1}(1-\tau_{t-1}) - c_{t-1}),$$

where b is a constant, a is assets, y is labour income, τ is an income tax rate, and c is consumption.

(a) Find the Euler equation.

(b) Find the consumption function.

(c) Suppose that τ is constant but that $y_t = \lambda y_{t-1} + \eta_t$, with $\mathrm{E}_{t-1}\eta_t = 0$. Specialize your answer to part (b) by replacing expectations with forecasts from this time series model. Assume that $\lambda \in (0,1)$.

(d) Assets are difficult to measure. So to deduce a test of this model, next write c_t in terms of c_{t-1}, y_t, and y_{t-1}.

(e) Find the variance of the change in c_t. Show how that variance depends on the persistence in shocks to labour income.

(f) Suppose that the fiscal authorities hope to increase consumption spending by reducing the income tax rate. Which will have a larger effect, a temporary tax reduction or a permanent one?

(g) Describe any differences between the effects of a tax cut that is announced in advance and one that comes as a surprise.

Answer

(a)

$$\mathrm{E}_t c_{t+1} = c_t$$

(b)

$$c_t = (\frac{r}{1+r})[a_t + \mathrm{E}_t \sum_{i=0}^{\infty} (1+r)^{-i} y_{t+i}(1-\tau_{t+i})]$$

(c)

$$c_t = (\frac{r}{1+r})[a_t + \frac{(1-\tau)y_t(1+r)}{(1+r-\lambda)}]$$

(d)

$$c_t = c_{t-1} + (\frac{r(1-\tau)}{1+r-\lambda})(y_t - \lambda y_{t-1})$$

(e)

$$\Delta c_t = (\frac{r(1-\tau)}{1+r-\lambda})\eta_t$$

So the variance of is the variance of η_t times the square of the term in brackets. Increases in λ increase this variance.

(f) By the same logic as in part (e) a permanent cut will have a larger effect.

(g) Whether temporary or permanent cut, PV of consumption change is PV of tax cut, so no effect on that. But if announced in advance then the effect starts now and is smoothed out in that sense.

7. Empirically, one finds that savings rates differ significantly across countries. This question explores how risk in returns to savings and also in income might affect the decision to save. To make this as simple as possible, consider a single saver who seeks to maximize expected utility over two periods: $u(c_1) + \mathrm{E}_1 u(c_2)$. Suppose that the utility function is of logarithmic form: $u(c_i) = \ln(c_i)$; $i = 1, 2$.

In period 1 $s_1 = y_1 - c_1$, while in period 2 $c_2 = y_2 + s_1(1 + r)$.

The consumer takes income and interest rates as given. Suppose that $y_1 = 1$, that y_2 can takes on one of two values: ϵ and $-\epsilon$, each with probability .5. Also suppose that r takes on one of two values: $.10 + \eta$ and $.10 - \eta$, each with probability .5. Thus risky income is modelled by using a large value of ϵ and risky returns on saving are modelled with a large value of η. The random variables η and ϵ are independent.

(a) Show the effect on saving of increasing income risk (ignore return risk).

(b) Show the effect on saving of increasing return risk (ignore income risk).

(c) Do your findings have any policy implications?

(d) Do the results in parts (a) and (b) depend on the assumption of log utility?

Answer

(a) First check on income risk. Use the budget equations in the Euler equation:

$$\frac{1}{1-s} = 1.10\left(\frac{.5}{-\epsilon + 1.10s} + \frac{.5}{\epsilon + 1.10s}\right)$$

See how increasing ϵ affects s. For $\alpha = 1$ (log utility) larger ϵ raises s (this is precautionary saving).

(b) Next check on interest risk. By the same method

$$\frac{1}{1-s} = \left[\frac{.5(1.10 + \eta)}{s(1.10 + \eta)} + \frac{.5(1.10 - \eta)}{s(1.10 - \eta)}\right]$$

Obviously there is no effect.

(c) To promote savings, add to income risk. Fiscal policy (such as tax timing) which reduces income risk will reduce savings.

(d) Under CRRA result (a) still holds, but result (b) depends on whether $\alpha >$ or < 1.

8. Suppose that consumption of nondurables and services can be described by the lcpih, with quadratic utility. Also ignore variation in interest rates and assume that the average interest rate equals the discount rate. Thus the Euler equation is: $\mathrm{E}_t c_{t+1} = c_t$.

Suppose that the budget constraint of a typical household can be written as:

$$a_t = (1 + r)(a_{t-1} + y_{t-1} - c_{t-1}),$$

where y is labour income and a is assets. Suppose that both c and y are measured in logs so that Δc, for example, is a growth rate.

(a) Find the consumption function, in terms of current and expected future labour income.

(b) Now suppose that labour income tends to evolve as follows:

$$y_t = \lambda y_{t-1} + \epsilon_t,$$

where $\epsilon_t \sim iid(0, \sigma^2)$. Describe a set of linear regressions that could be used to test the theory.

(c) Suppose that $\lambda = 1$. Find the ratio of the variance of consumption growth to the variance of income growth. Does this prediction match the empirical evidence for most countries?

(d) Does empirical evidence suggest that one can ignore variation in interest rates in describing variation in consumption growth over time?

Answer

(a) The consumption function is:

$$c_t = (\frac{r}{1+r})[a_t + \mathrm{E}_t \sum_{i=0}^{\infty} (1+r)^{-i} y_{t+i}]$$

which gives

$$c_t = c_{t-1} + (\frac{r}{1+r})[\sum_{i=0}^{\infty} (1+r)^{-i} (\mathrm{E}_t y_{t+i} - \mathrm{E}_{t-1} y_{t+i})]$$

(b)

$$c_t = c_{t-1} + (\frac{r}{1+r-\lambda})(y_t - \lambda y_{t-1})$$
$$y_t = \lambda y_{t-1}$$

Then discuss identification.

(c) If $\lambda = 1$ then $var\Delta y = \sigma^2$.

$$\Delta c_t = (\frac{r}{1+r-\lambda})(y_t - y_{t-1}) = \epsilon_t,$$

so the variance ratio is 1. In fact, we find ratios less than 1 for most countries (consumption is smoother than income; see section 1). So the lcpih faces an excess smoothness puzzle.

(d) Hall evidence (and similarly for Canada) suggests that there is very low intertemporal elasticity of substitution (Leontief-type indifference curves) so that ignoring r may not matter. But there is some limited support for consumption-based asset pricing models.

9. An election is expected and the authorities hope to stimulate consumption spending by cutting taxes. They want to know whether to cut taxes in both periods (of a two-period model) or to announce that only the first-period tax rate will be lower. One pundit argues that the effect will be greatest if people know that the tax cut is temporary, because then they will concentrate their spending. Another claims that a permanent cut will have a much larger effect. You are to advise them on which plan will be most expansionary.

Consider a two-period model of household consumption spending:

$$\max U = (c_1 - c_1^2/2) + \beta(c_2 - c_2^2/2)$$

subject to

$$c_1 + c_2/(1+r) = y(1-\tau_1) + y(1-\tau_2)/(1+r),$$

so that income is the same in each period. Assume that $\beta(1+r) = 1$. In what follows, we shall use this budgetting problem to find the effects of the two different tax cut proposals, taking income and interest rates as given (they could be affected in general equilibrium since this is not an endowment economy, but we shall ignore such effects).

(a) Find the Euler equation linking c_1 and c_2.

(b) Use the Euler equation and the budget constraint to solve for c_1 in terms of incomes and the interest rate.

(c) Will c_1 increase more if t_1 is reduced or if both t_1 and t_2 are reduced?

Answer

(a) The Euler equation is: $c_1 = c_2$.

(b) Using the consumption function:

$$c_1 = [\frac{1+r}{2+r}] \cdot \big[y(1-\tau_1) + y(1-\tau_2)/(1+r)\big]$$

(c)

$$dc_1/d\tau_1 = [\frac{1+r}{2+r}](-y) < 0$$

$$dc_1/d\tau_2 = [\frac{1+r}{2+r}] \cdot (-y)[1/(1+r)] < 0$$

So a permanent cut will be more expansionary; from the permanent income hypothesis. But in a two-period model if g does not fall then t_2 must rise when t_1 falls; so one could also take that into account.

10. Show that for CRRA utility the coefficient of relative risk aversion equals the inverse of the intertemporal elasticity of substitution.

Answer

First,

$$\eta = -\frac{\Delta \ln(c_t)}{\Delta \ln(p_t)},$$

because elasticities are positive. The Euler equation (without uncertainty) is:

$$c_{t-1}^{-\alpha} = \frac{\beta p_{t-1} c_t^{-\alpha}}{p_t},$$

where $p_{t-1}/p_t = 1 + r_{t-1}$. Thus

$$\alpha \ln(\frac{c_t}{c_{t-1}}) = \ln(\beta) - \ln(\frac{p_t}{p_{t-1}})$$

so that, taking differences, $\eta = 1/\alpha$.

11. Studying consumption functions usually requires some assumptions about labour income. Suppose that households can budget with a constant interest rate such that $1 + r = 1/\beta$, where β is their discount factor. Also suppose that they have quadratic utility so that

$$E_t c_{t+1} = c_t.$$

They face a budget constraint given by

$$a_t = (1 + r)(a_{t-1} + y_{t-1} - c_{t-1}),$$

where c is consumption, y is labour income, and a is wealth.

(a) Solve for current consumption in terms of lagged consumption and current and expected future labour income.

(b) Now suppose that

$$y_t = \mu \cdot t + \epsilon_t,$$

where ϵ_t is distributed $iid(0, \sigma^2)$. Thus labour income has a time trend. Replace expectations in your answer to (a) with forecasts from this description of labour income, and so solve for c_t in terms of observable variables.

(c) Does empirical evidence support this model?

Answer

(a)

$$c_t = c_{t-1} + \frac{r}{1+r}\sum_{j=0}^{\infty}(1+r)^{-j}[E_t y_{t+j} - E_{t-1}y_{t+j}]$$

(b)

$$c_t = c_{t-1} + \frac{r}{1+r}\epsilon_t$$

(c) There is evidence against the Euler equation by itself. Then there is excess sensitivity and excess smoothness

12. Perhaps rejections of the permanent-income hypothesis for aggregate consumption are due to our misrepresenting expectations. Suppose that the interest rate is constant and that $\beta(1+r) = 1$. There are two time periods, denoted 1 and 2. The representative agent maximizes:

$$EU = -(a - c_1)^2 - \beta E_1(a - c_2)^2$$

subject to

$$c_1 + \frac{E_1 c_2}{1+r} = y_1 + \frac{E_1 y_2}{1+r}.$$

(a) Derive the Euler equation linking consumption expenditures in the two time periods.

(b) Solve for consumption functions for both c_1 and c_2.

(c) Now suppose that $E_1 y_2 = 0.6 y_1$. Show that an investigator who overestimates the persistence in income and assumes $E_1 y_2 = 0.8 y_1$ will find apparent 'excess sensitivity' of c_2 to y_1.

Answer

(a)

$$E_1 c_2 = c_1$$

(b)

$$c_1 = \frac{1+r}{2+r}(y_1 + \frac{E_1 y_2}{1+r})$$

Then

$$c_2 = c_1 + \epsilon$$

where ϵ is the income surprise.

(c) We know that

$$c_2 = c_1 + \epsilon = c_1 + (y_2 - .6y_1).$$

Thus

$$c_2 = c_1 + (y_2 - .8y_1) + .2y_1$$

which will look like excess sensitivity to the investigator.

13. Imagine that a household derives utility from a stock of durable goods, denoted d_t, according to:

$$E_t \sum_{i=0}^{\infty} (1+\theta)^{-i} (\lambda d_{t+i} - \gamma d_{t+i}^2),$$

where λ and γ are constants. The stock of durables evolves according to:

$$d_t = (1-\delta) d_{t-1} + c_t,$$

where δ is the depreciation rate, and c_t is expenditure on durables. The household's assets evolve according to

$$a_{t+1} = (1+r_t)(a_t + y_t - c_t),$$

where y_t is labour income.

(a) State the first-order conditions for an optimum.

(b) Suppose that $r = \theta$ and that $\delta = 0$. What are the time series properties of d_t and c_t?

(c) Can durability perhaps account for evidence of excess smoothness in aggregate consumption expenditures?

Answer

(a)

$$E_t (\frac{1+r}{1+\theta})(\lambda - 2\gamma d_{t+1}) = \lambda - 2\gamma d_t$$

plus the budget constraints.

(b) Now d_t follows a random walk, and c_t is white noise.

(c) I don't think so. First, most tests use consumption of nondurables and services. Second, although certainly c_t will have lower variance than in the usual model, suppose that

$$y_t = \rho y_{t-1} + \nu_t,$$

as in the notes. Then recognize that c_t is now the innovation in marginal utility. So

$$c_t = (\frac{r}{1+r-\rho})(y_t - \rho y_{t-1}),$$

(what is called ϵ_t in the notes). Suppose that $\rho = 1$. Then $var\Delta d_t = var c_t = var\Delta y_t$. If c_t is white noise then the variance of $c_t - c_{t-1}$ will be *larger* than the variance of $c_t = \nu_t$. So this cannot explain findings of excess smoothness.

14. Detailed tests of the permanent income hypothesis involve statistical evidence on both consumption, c_t, and labour income, y_t. Suppose that the Euler equation is:

$$c_t = \mathrm{E}_t c_{t+1},$$

and that the budget constraint is:

$$a_t = (1 + r)(a_{t-1} + y_{t-1} - c_{t-1}).$$

Finally, suppose that labour income evolves according to:

$$\Delta y_t = \mu + \epsilon_t, \quad \epsilon_t \sim iid(0, \sigma^2).$$

(a) Solve for the consumption function which relates consumption to current income and lagged consumption and income (*i.e.* and does not require observations on assets).

(b) Describe the cross-equation restrictions which could be used to identify parameters and test the model.

(c) Describe how this framework could be used to test for excess sensitivity and excess smoothness.

Answer

(a) The consumption function is:

$$\Delta c_t = \Delta y_t - \mu.$$

(b) The restrictions are simple then. There is a common μ in both equations. The overidentification allows a test. The two equations should have the same innovation variance, too.

(c) To test for excess sensitivity one would include y_{t-1} separately in the consumption equation. To test for excess smoothness, one would compare the variances of Δc_t and Δy_t, which should be equal here because all income changes are permanent.

15. Suppose that a typical household's assets, a_t, evolve as follows:

$$a_t = (1 + r)(a_{t-1} + y_{t-1} - c_{t-1}),$$

where y_t is labour income and c_t is consumption. The household tries to set consumption so that:

$$c_t = E_t c_{t+1},$$

i.e. the random walk model applies.

(a) Find the consumption function in terms of current assets and current and expected future labour income.

(b) Suppose that income evolves as follows:

$$y_{t+1} = y_t + \epsilon_{t+1},$$

where ϵ_{t+1} is unpredictable and has mean zero. Find the predicted coefficients in the regression of c_t on a_t and y_t.

(c) Suppose that measurements on a_t are not available. How could you test the consumption function with aggregate data?

(d) Does empirical evidence support this model?

(e) Recently in Canada aggregate consumption expenditures have grown very slowly, despite low interest rates. Is that perplexing given macroeconomic theory?

Answer

(a)

$$c_t = \frac{r}{1+r}[a_t + y_t + E_t \frac{y_{t+1}}{1+r} + ...]$$

(b)

$$c_t = \frac{r}{1+r} a_t + y_t$$

This makes sense because all income changes are permanent.

(c) One finds that:

$$c_t - c_{t-1} = y_t - y_{t-1}.$$

This could be tested easily by regression methods, perhaps jointly with the y-equation.

(d) Not very well. First, in tests of the random walk model there seems to be a role for lagged income and perhaps other variables. Second, the variance of consumption growth is less than the variance of income growth, even though most changes in labour income are relatively permanent (as they are in this question).

(e) The standard Irving Fisher diagram predicts that high interest rates are associated with postponing consumption. But estimates of the intertemporal elasticity of substitution are very small, at least for consumption expenditures on non-durables and services. Given this statistical evidence, it isn't surprising that consumption has not responded much to interest rates. The small response of durables expenditure is more difficult to explain though. So is the overall small response given that there are liquidity constraints on some households. In the model in this question, low consumption spending is explained by low permanent labor income. What has happened to labor income in Canada?

16. Imagine an economy with an infintely-lived, representative agent. At time 0 the agent maximizes:

$$\sum_{t=0}^{\infty} \beta^t E_0(c_t - bc_t^2),$$

subject to

$$c_t + k_{t+1} = y_t = Ak_t + u_t,$$

where u_t is a random endowment shock. Assume that $A = \beta^{-1}$.

(a) Find the optimal decision rule for c_t

(b) Find the law of motion for the capital stock, k_t.

(c) Suppose now that the endowment follows a first-order autoregression:

$$u_t = \rho u_{t-1} + \epsilon_t$$

where ϵ_t is white noise. Specialize your answers to parts (a) and (b) for this case.

(d) Graph the impulse response functions for output and consumption in response to permanent ($\rho = 1$) and temporary ($\rho = 0$) endowment shocks.

(e) Is there any justification for the production function assumed in this model?

Answer

(a) Let $r = A - 1$.

$$c_t = rk_t + \frac{r}{1+r}\sum_{i=0}^{\infty}(1+r)^{-i}E_t u_{t+i}$$

(b)

$$k_{t+1} = k_t + u_t - \frac{r}{1+r}\sum_{i=0}^{\infty}(1+r)^{-i}E_t u_{t+i}$$

(c)

$$c_t = rk_t + u_t(\frac{r}{1+r-\rho})$$

and

$$k_{t+1} = k_t + u_t(1 - \frac{r}{1+r-\rho})$$

(d) [graphs]

(e) This is obviously the Ak model, so it can perhaps be justified if we assume that k includes human capital.

17. This question examines the implications of a simple model of portfolio choice. Suppose that a representative agent has the following utility function:

$$E_0 \sum_{t=0}^{\infty} \beta^t ln(c_t),$$

with $0 < \beta < 1$. This is maximized subject to

$$w_{t+1} = R_{t+1}(w_t - c_t),$$

with w_0 given. Here c_t is consumption, w_t is wealth, and R_{t+1} is the gross return on investment between time t and time $t+1$. Wealth satisfies the transversality constraint

$$lim_{t\to\infty} E_0 \beta_t w_t = 0.$$

The return series is independently and identically distributed over time, and is given exogenously.

(a) Find the optimal consumption policy.

(b) Describe the theoretical restrictions on the bivariate, first-order autoregression in $\{ln(w_t), ln(c_t)\}$.

(c) Wealth is difficult to measure. Find theoretical restrictions on the univariate time series process for $ln(c_t)$. Briefly interpret your result.

(d) One way to make this a general equilibrium model – so that the return is no longer exogenous – is to assume that there is a non-storable dividend series $\{d_t\}$ and that $c_t = d_t$. Suppose that a claim to this dividend series has price p_t. Find the equilibrium price-dividend ratio in this economy.

(e) Prove that the average equity premium is positive in the economy of part (d).

Answer

(a)

$$c_t = (1-\beta)w_t$$

(b)

$$\begin{aligned} ln(w_t) &= ln(\beta) + ln(w_{t-1}) + ln(R_t) \\ ln(c_t) &= ln[\beta(1-\beta)] + ln(w_{t-1}) + ln(R_t) \end{aligned}$$

(c) Clearly

$$ln(c_t) = ln(c_{t-1}) + ln\beta + lnR_t,$$

the random walk model.

(d)

$$\frac{p_t}{d_t} = \frac{\beta}{1-\beta}$$

(e) The gross riskless rate is

$$R_{bt+1} = \frac{1}{\beta d_t E_t \frac{1}{d_{t+1}}}$$

The expected return on the fruit tree is

$$R_{t+1} = \frac{E_t d_{t+1}}{\beta d_t}$$

which is larger, by Jensen's inequality.

18. This question studies the quadratic model used in much recent empirical research on aggregate consumption. Suppose that identical households seek to maximize:

$$E_0 \sum_{t=0}^{\infty} \beta^t (c_t - bc_t^2),$$

subject to

$$a_t = (1 + r)a_{t-1} + y_t - c_t.$$

Here a_t is assets, y_t is labour income, and c_t is consumption. Assume that $\beta = 1/(1 + r)$.

(a) State the Euler equation for this problem. Briefly describe how this could be used to test the theory.

(b) Suppose that labour income seems to follow a random walk:

$$y_t = y_{t-1} + \nu_t.$$

Here ν_t has mean zero and variance σ^2 and is unforecastable. Derive the consumption function predicted by the theory and show how it could be tested without data on assets.

(c) Does this model make a realistic prediction for the variance of consumption relative to labour income?

(d) Is there any reason to include an interest rate in the regression you found in part (b)? Would including an interest rate improve the predictions of the theory?

Answer

(a)

$$c_t = E_t c_{t+1}$$

This is the random walk model, and it could be tested by regressing the change in consumption on other variables to see if that change can be forecasted.

(b) Solving the budget constraint forward and using the Euler equation and the random walk in income gives:

$$c_t = ra_t + y_t.$$

Then to avoid measuring a_t, lag this, multiply by $1 + r$ and subtract to give:

$$c_t = c_{t-1} + y_t - y_{t-1}.$$

This could be tested using a linear regression, with restrictions on the joint process for c and y.

(c) No. In practice we find the the change in consumption has a smaller variance than the change in labour income.

(d) The interest rate could be included (i) from the Euler equation (as in the regressions of Wirjanto) or (ii) to reflect liquidity constraints. The empirical evidence suggests a relatively small effect of interest rates on consumption ex durables though.

19. The quadratic version of the permanent income hypothesis makes some strong assumptions. In this question we explore how tests of that model would be affected by changing one of those assumptions. Suppose that all households maximize:

$$E_0 \sum_{t=0}^{\infty} (\frac{1}{1+\theta})^t u(c_t),$$

subject to

$$a_t = (1+r)a_{t-1} + y_t - c_t,$$

and a_0 given. The utility function is:

$$u(c_t) = c_t - \frac{c_t^2}{2}.$$

Do *not* assume that $\theta = r$.

(a) State the Euler equation for consumption.

(b) Could you estimate θ from the Euler equation?

(c) Is Δc_{t+1} unpredictable at time t?

(d) Does empirical evidence support the quadratic version of the permanent income hypothesis?

Answer (a)

$$1 - c_t = E_t \frac{1+r}{1+\theta}(1 - c_{t+1}).$$

(b) Rewriting:

$$c_{t+1} = \frac{r-\theta}{1+r} + \frac{1+\theta}{1+r} c_t + \epsilon_{t+1}$$

so with two parameters both θ and r can be identified. Notice that you do not need to find a consumption function, and indeed no income law of motion is given.

(c) You can see that Δc_{t+1} is autocorrelated, so it would appear that the model with $\theta = r$ is wrong, even though the more general version is correct here.

(d) [Discussion of rejections of the first-order condition and of solved consumption functions.]

20. This question studies the role of bequests in explaining savings. Suppose that agents live for only one period. They value their own consumption and the utility of their descendants (each agent has one descendant). The utility function of someone who lives in generation t is

$$U_t = \ln(c_t) + \gamma U_{t+1},$$

where γ is the weight placed on the utility of the descendant. In generation t, agents receive an endowment y_t and a bequest b_t left to them by their ancestors. They can divide these resources into consumption c_t and a bequest to their descendants, b_{t+1}. Thus the budget constraint is:

$$b_t + y_t = c_t + \frac{b_{t+1}}{1+r}.$$

Finally, bequests cannot be negative: $b_t \geq 0$.

(a) Suppose that $r = 0.2$ (this is fixed by the storage technology) and $\gamma = 0.5$. There is no uncertainty. Suppose that y_t is a constant, y. What is the optimal consumption plan?

(b) If the government announces that it is lowering y_t and raising y_{t+1} (by collecting taxes now rather than later) will national saving be affected?

Answer (a) To see how much to leave as a bequest, the current generation must consider its effect on their descendant's utility. But in turn that depends on how much of the bequest the descendant consumes. This reasoning leads us to substitute for U_{t+1} and so on, to get:

$$U_t = \ln(c_t) + \gamma \ln(c_{t+1}) + \gamma^2 \ln(c_{t+2}) + \dots$$

subject to

$$b_{t+1} = (1+r)(b_t + y - c_t),$$

which is a deterministic dynamic programming problem. The repeated substitution shows that this is effectively a single-agent model, with discount factor γ. Also, the original equation resembles Bellman's equation (with U playing the role of V). Guessing that $c_t = k_0 + k_1 b_t$ and using this in the Euler equation gives:

$$k_0 + k_1 b_t = \frac{k_0 + k_1(1.2)(y - k_0)}{.6} + \frac{k_1(1.2)(1-k_1)}{.6} b_t.$$

Thus $k_0 = y/3.5$ and $k_1 = 0.5$.

(b) Note first that the discount rate θ is 1, because $\gamma = 0.5$. However, Ricardian equivalence does not require that $\theta = r$. Thus, national saving won't be affected as long as the corner solution with a zero-bequest is not reached. If the increase in taxes is so large that the optimal bequest goes to zero, then a further increase in taxes will raise national saving and Ricardian equivalence won't hold.

21. This question studies the predictions of the intertemporal approach to the current account, *i.e.* the idea that we can describe the current account using models of saving and investment. Consider a small, open economy, which faces a constant, world interest rate, r. Its external debt b_t evolves this way:

$$b_t = (1+r)(b_{t-1} - nx_{t-1}),$$

where net exports, nx_t, are given by $nx_t = y_t - c_t$. The country receives a nonstorable endowment y_t, which follows this stochastic process:

$$y_t = \rho y_{t-1} + \nu_t,$$

where ν_t is white noise. Agents in the domestic economy are identical, and can be represented with the utility function:

$$E \sum_{t=0}^{\infty} \beta^t \big(c_t - \frac{c_t^2}{2}\big).$$

(a) State the Euler equation describing domestic consumption decisions and describe how it could be used to test the model and to estimate β. Explain how and why differences between β and $1/(1+r)$ are reflected in the path of consumption.

(b) Now assume that $\beta = 1/(1+r)$. According to the theory, how is the current account related to national income and to foreign debt?

(c) How would 'excess smoothness' in consumption be reflected in the behaviour of the current account? How could you test for this?

Answer (a) The Euler equation is:

$$E_t c_{t+1} = \left(1 - \frac{1}{\beta(1+r)}\right) + \frac{1}{\beta(1+r)} c_t.$$

The values of β and r could be estimated from two OLS regression coefficients. Any difference between β and $1/(1+r)$ is reflected in 'consumption tilting'. For example, if the country is very patient, then β is small, and the intercept will be small and slope large in this regression. The country will be a lender and have faster consumption growth.

(b) The consumption function is:

$$c_t = \frac{r}{1+r}(-b_t) + \frac{r}{1+r-\rho} y_t.$$

The current account is:

$$ca_t = b_{t+1} - b_t = nx_t + rb_t.$$

Combining this definition with the consumption function gives:

$$ca_t = (1+r)(y_t - \frac{r}{1+r-\rho}y_t) = \frac{(1+r)(1-\rho)y_t}{1+r-\rho}.$$

(c) Some empirical evidence suggests that consumption does not adjust to changes in income as much as the theory predicts, given the persistence in those changes. But that means net exports will be more responsive (procyclical) than the theory predicts. The theory could be tested using the equation in part (b) and the income process, to see whether the ρ in income is the same as the one in the current account equation; under excess smoothness the latter will be smaller.

22. Debt bubbles: Consider the government's real budget constraint:

$$E_{t-1}b_t = (1+r)b_{t-1} - E_{t-1}s_t$$

Show that

$$b_t = E_t \sum_{i=1}^{\infty}(1+r)^{-i}s_{t+i} + k_t(1+r)^t$$

is a solution to the difference equation provided that $E_{t-1}k_t = k_{t-1}$.

23. Revisions in expectations: Suppose that an exogenous variable evolves as follows:

$$z_t = kz_{t-1} + \nu_t$$

where $E(z_{t-1}\nu_t) = 0$. Consider an information set I_t which includes z_t.

(a) Find $E(z_{t+1}|I_t)$. Find $E(z_{t+2}|I_t)$.

(b) Suppose that an endogenous variable m_t is given by

$$m_t = z_{t+1} + \gamma z_{t+2}.$$

Find $E(m_t|I_{t-1})$.

(c) Find $E(m_t|I_t)$. Hence find the revision in expectations as a result of information arriving between $t-1$ and t.

24. Consider an infinitely-lived government which solves the following problem of setting tax rates:

$$\min E_0 \sum_{t=0}^{\infty}(1+r)^{-t}(a - b\tau_t)^2$$

subject to $t_t = \tau_t \cdot y_t$; $s_t = t_t - g_t$; $b_t = (1+r)(b_{t-1} - s_{t-1})$

in which τ_t is the proportional tax rate, t_t is total revenue, y_t is income or output, g_t is government spending (exclusive of interest payments), s_t is the government surplus, b_t is government debt, and r is a constant interest rate. All variables are real.

(a) Find the Euler equation linking tax rates in adjacent periods.

(b) Suppose that $\{y_t\}$ is constant. Suppose that

$$g_t = \rho g_{t-1} + v_t$$

with $0 < \rho < 1 + r$. Find the response of τ_t to a shock v_t to government spending.

(c) If ρ is large, then shocks are relatively permanent while if ρ is small they are relatively temporary. Do tax rates respond more to a permanent or to a temporary shock?

Answer

(a)

$$\text{Minimize} \quad E_0 \sum_{t=0}^{\infty} (1+r)^{-t} (a - b\tau_t)^2$$

subject to $t_t = \tau_t \cdot y_t$, $s_t = t_t - g_t$ and $b_t = (1+r)(b_{t-1} - s_{t-1})$.
Find the Euler equation.
Combining the constraints we get

$$b_t = (1+r)(b_{t-1} + g_{t-1} - \tau_{t-1} \cdot y_{t-1}).$$

Solving it forward yields

$$\sum_{j=0}^{\infty} \frac{\tau_{t+j} \cdot y_{t+j}}{(1+r)^j} = b_t + \sum_{j=0}^{\infty} \frac{g_{t+j}}{(1+r)^j}$$

The Lagrangian is

$$\mathrm{L} \quad = E_t \sum_{j=0}^{\infty} \left\{ \left(\frac{1}{1+r} \right)^j (a - b\tau_{t+j})^2 + \lambda \left[b_t + \sum_{j=0}^{\infty} \frac{g_{t+j}}{(1+r)^j} - \sum_{j=0}^{\infty} \frac{\tau_{t+j} \cdot y_{t+j}}{(1+r)^j} \right] \right\}$$

The FOC with respect to τ_{t+j} is

$$\frac{\partial \mathrm{L}}{\partial \tau_{t+j}} = E_{t+j} \left\{ \left(\frac{1}{1+r} \right)^j (-2b)(a - b\tau_{t+j}) - \frac{\lambda y_{t+j}}{(1+r)^j} \right\} = 0$$

$$\Rightarrow \lambda = -2b E_{t+j} \frac{(a - b\tau_{t+j})}{y_{t+j}} = -2b \frac{(a - b\tau_{t+j})}{y_{t+j}}$$

Similarly,

$$\frac{\partial \mathrm{L}}{\partial \tau_{t+j+1}} \Rightarrow \lambda = -2b E_{t+j} \frac{(a - b\tau_{t+j+1})}{y_{t+j+1}}$$

Eliminating λ gives

$$\frac{(a-b\tau_{t+j})}{y_{t+j}} = E_{t+j}\frac{(a-b\tau_{t+j+1})}{y_{t+j+1}}$$

$$\Rightarrow (a-b\tau_{t+j}) = E_{t+j}(a-b\tau_{t+j+1})\frac{y_{t+j}}{y_{t+j+1}}$$

Or, setting $j=0$ we have

$$(a-b\tau_t) = E_t(a-b\tau_{t+1})\frac{y_t}{y_{t+1}} \qquad (*)$$

If in addition there is no growth in output (as in part (b)) then the answer simplifies to:

$$\tau_t = E_t\tau_{t+1}$$

(b) Then:

$$\tau_t = y^{-1}(1+r)^{-1}r(b_t + g_t + E_t g_{t+1}/(1+r) + ...)$$

So that

$$\tau_t - \tau_{t-1} = \epsilon_t = \left(\frac{r}{(1+r)y}\right)\sum_{j=0}^{\infty}(1+r)^{-j}(E_t g_{t+j} - E_{t-1}g_{t+j})$$

$$= v_t(r/y)/(1+r-\rho)$$

(c) Call $k = (r\rho)/(1+r-\rho):\ dk/d\rho = (1+r-\rho)^{-2}[(1+r-\rho)r - r\rho(-1)] > 0$

So the more permanent the change in g the larger the jump in the tax rate in response to the news.

25. Predicting the effects of changes in macroeconomic policy may require one to study policy rules (*i.e.* complete paths for policy variables). To study this idea, consider the representative consumer's problem in a two-period model. Preferences are represented by:

$$U = logc_1 + \beta logc_2.$$

The budget constraint is

$$c_1 + c_2/(1+r) = (y_1 - t_1) + (y_2 - t_2)/(1+r),$$

where c_1 is consumption in period 1, c_2 is consumption in period 2, y is income and t is a tax.

(a) Find the Euler equation relating c_1 and c_2.

(b) Solve for the consumption function.

(c) Show the effect (if any) on consumption of a pure change in the timing of taxes, involving a tax cut in period 1, with no changes in government spending. The government's budget constraint is

$$g_1 + g_2/(1+r) = t_1 + t_2/(1+r).$$

(d) Now suppose that these two-period lived consumers are born in overlapping generations, with no population growth, and that the economy and the government go on forever. Show that changes in the timing of taxes now may affect the interest rate.

Answer

(a) $c_1 = c_2/[\beta(1+r)]$

(b) $c_1 = (1+\beta)^{-1}[(y_1 - t_1) + (y_2 - t_2)/(1+r)]$

(c) From the government's budget constraint:

$$0 = dt_1 + dt_2/(1+r)$$

Hence there is no effect, since $t_1 + t_2/(1+r)$ is unchanged.

(d) Simple to show in OLG ...

26. Some economists have suggested that the best way to explain the historical pattern of deficits is to observe that governments may try to smooth tax rates (and revenue) over time. This question studies the pattern of deficits that would result from such behaviour. Suppose that the real interest rate is determined by technology as r. Consider a two-period model. Let t be revenue and g be expenditure, with subscripts denoting the two time periods. Suppose that the government's preferences are to choose t_1 and t_2 to minimize:

$$L = -(a - t_1)^2 - (1+r)^{-1}E_1(a - t_2)^2$$

subject to its budget constraint

$$t_1 + E_1 t_2/(1+r) = g_1 + E_1 g_2/(1+r),$$

and taking spending as given. Note that t_1 is set based on $E_1 g_2$ but that g_2 is known when t_2 is set.

(a) Find the Euler equation linking tax revenues over time.

(b) Combine the Euler equation with the budget constraint to give the optimal tax revenues in each time period.

(c) Suppose that spending follows the pattern: $g_2 = \lambda g_1 + \epsilon_2$, with $E_1\epsilon_2 = 0$. Specialize your answer to part (b).

(d) Show that tax revenue responds more to a persistent or relatively permanent change in government spending than to a temporary one.

(e) Does historical evidence for Canada provide any support for this general view?

Answer

(a) The Euler equation is $E_1 t_2 = t_1$.

(b) The functions are:

$$
\begin{aligned}
t_1 &= [(1+r)/(2+r)] \cdot [g_1 + E_1 g_2/(1+r)] \\
t_2 &= (1+r)g_1 + g_2 - (1+r)t_1 \\
&= (1+r)g_1 + g_2 - (1+r)[(1+r)/(2+r)] \cdot [g_1 + E_1 g_2/(1+r)] \\
&= (1+r) - [(1+r)^2/(2+r)]g_1 + g_2 - [(1+r)/(2+r)]E_1 g_2 \\
&= g_1(1+r)[1-(1+r)/(2+r)] + g_2 - E_1 g_2[(1+r)/(2+r)]
\end{aligned}
$$

(c) If $g_2 = \lambda g_1 + \epsilon$ then $E_1 g_2 = \lambda g_1$.
Thus the answer in (b) becomes:

$$
\begin{aligned}
t_1 &= [(1+r)/(2+r)] \cdot [g_1 + \lambda g_1/(1+r)] \\
t_2 &= g_1(1+r)[1-(1+r)/(2+r)] + \lambda g_1 + \epsilon - \lambda g_1[(1+r)/(2+r)]
\end{aligned}
$$

(d) $dt_1/d\lambda > 0$.

(e) Rough correspondence I guess.

27. Presumably the market value of a country's external debt (both private and public) depends on expectations of its future trade surpluses. For example, foreign investors might be reluctant to hold debt of a country which is expected to run large trade deficits because they fear those deficits will lead to a depreciation, which would erode the value of their holdings. External debt (b_t) and the trade surplus (s_t) are linked through the identity:

$$b_t = (1+r)(b_{t-1} - s_{t-1}).$$

(a) Solve this equation forwards, using the expectations operator because future surpluses are unknown.

(b) Suppose that the surplus follows this time series process:

$$s_t = \mu + s_{t-1} + \epsilon_t,$$

where ϵ_t is an iid shock with mean zero. This holds that changes in the trade surplus have mean μ. Assume that debt-holders forecast with this pattern and use it to replace the expectations in your answer to part (a).

(c) Would it be accurate to say that economic theory predicts that the market value of external debt should be very sensitive to small changes in the average change in the trade balance?

Answer

(a) $b_t = s_t + E_t s_{t+1}/(1+r) + ...$ with transversality

(b)

$$\begin{aligned}
b_t &= s_t + (s_t + \mu)/(1+r) + (s_t + 2\mu)/(1+r) + ... \\
&= s_t(1 + 1/(1+r) + ...) + \mu/(1+r) + 2\mu/(1+r) + ... \\
&= s_t(1+r)/r + \mu/r + \mu/r(1+r) + ... \\
&= s_t(1+r)/r + (\mu/r) \cdot (1+r)/r \\
&= s_t(1+r)/r + \mu(1+r)/r^2
\end{aligned}$$

(c) Yes. Suppose that $r = 0.05$. Then $db_t/d\mu = 420$.

28. To study the predicted effect of government spending on interest rates, consider a two-period-lived OLG model. Agents maximize:

$$\ln(c_{1t}) + \beta \ln(c_{2t+1}),$$

subject to

$$c_{1t} + \frac{c_{2t+1}}{1+r} = y_t - \tau_t.$$

Output y_t is owned by young agents, and they pay a tax τ_t. Old agents consume their savings. The government balances its budget each period, setting

$$g_t = \tau_t.$$

(a) Solve for the individual agent's consumption function.

(b) Suppose that output and government spending are growing as follows:

$$\begin{aligned}
y_t &= (1+\mu)y_{t-1} \\
g_t &= (1+\mu)g_{t-1}.
\end{aligned}$$

Solve for the interest rate, r, in a competitive equilibrium.

(c) What is the effect on the interest rate of an unexpected, temporary increase in government spending?

Answer

(a)

$$\begin{aligned}
c_{1t} &= \frac{1}{1+\beta}(y_t - g_t) \\
c_{2t+1} &= \frac{\beta}{1+\beta}(1+r)(y_t - g_t)
\end{aligned}$$

(b) From market clearing,

$$c_{1t} + c_{2t} + g_t = y_t.$$

That gives $r = \mu$.

(c) Take the benchmark in which $\mu = 0$:

$$\frac{y-g}{1+\beta} - \frac{\epsilon}{1+\beta} + \frac{\beta}{1+\beta}(1+r)(y-g) + g + \epsilon = y.$$

It looks like r falls temporarily. Then in the next period it rises; the intuition is that young saving falls. Then in the period after that it returns to μ.

29. Some economists have argued that tax smoothing provides an accurate description of government budget deficits over time. Suppose that the government sets tax revenues t_t so that:

$$\mathrm{t}_t = E_t \mathrm{t}_{t+1}$$

and that its budget constraint is

$$b_t = (1+r)b_{t-1} + g_t - \mathrm{t}_t.$$

(a) Find the optimal tax revenue as a function of the current debt and expected future levels of spending.

(b) Suppose that spending evolves as follows:

$$g_t = \rho g_{t-1} + \nu_t,$$

with $0 < \rho < 1$. Let s_t denote the primary surplus. Derive rational expectations cross-equation restrictions on $\{g_t, s_t\}$ that can be used to test the tax smoothing hypothesis.

(c) What would be meant by 'excess smoothness' of tax revenue?

30. Consider the following three-period government budget constraint:

$$t_1 + E_1 \frac{t_2}{1+r} + E_1 \frac{t_3}{(1+r)^2} = g_1 + E_1 \frac{g_2}{1+r} + E_1 \frac{g_3}{(1+r)^2},$$

where r is fixed.

(a) Use the tax-smoothing hypothesis to solve for t_1 if $g_1 = E_1 g_2 = E_1 g_3 = 4$.

(b) Find the time paths of tax revenue and the primary deficit if government spending in period 2 rises to 6 permanently and unexpectedly. Compare this to the case where the increase is temporary.

(c) Are the predictions of the tax-smoothing hypothesis supported by empirical evidence?

Answer

(a) $t_1 = 4$

(b) In the case of a permanent increase, $t_2 = t_3 = 6$, and the deficit is zero in each period. In the case of a temporary increase,

$$t_2 = t_3 = (\frac{1+r}{2+r})(6 + \frac{4}{1+r}).$$

Here the deficit is zero in the first period, positive in the second period, and there is a surplus in the third period.

(c) The prediction that temporary increases in spending (as in wars) are financed largely by deficits does seem to be supported historically. There is less evidence of tax-smoothing in the case of *anticipated* spending changes.

31. This question uses a simple equilibrium model to study the effects of fiscal policy on the current account and interest rates. Suppose that the world consists of two countries, indexed by i. There are many time periods. In each country, households seek to maximize:

$$U_i = E_0[\ln(c_{i1}) + \beta \ln(c_{i2})].$$

In each country there is a random endowment of a single, non-storable endowment, y_{it}. For each time period and country this random variable has a mean of one, and it is uncorrelated across countries.

(a) In a competitive equilibrium what will be the correlation between c_1 and c_2?

(b) Now suppose that there is a government in country 1, which spends g_1 in period 1 and g_2 in period 2. In a competitive equilibrium, describe the effects of government spending on the current account and the world interest rate.

(c) A statistician is studying the effect of fiscal policy on international borrowing and lending. She runs a cross-section regression of the current account deficit on government spending. Would you expect the coefficient in this regression to be the same for large and small economies? Should any other variables be included?

Answer

(a) The correlation will be one.

(b) Now market clearing requires

$$c_{1t} + c_{2t} + g_t = y_{1t} + y_{2t}.$$

The resource constraint for country 1 is

$$c_{11} + g_1 + \frac{c_{12} + g_2}{1+r} = y_{11} + \frac{y_{12}}{1+r}.$$

It is clear that a large g_1, for example, will lead to a current account deficit and raise the interest rate.

(c) Clearly the coefficient should be larger for small economies. They do not affect world interest rates. A fiscal expansion in a large economy will raise the interest rate, discouraging borrowing.

Technology shocks (or other influences on output) should also be included.