1. INTRODUCTION

> *But the age of chivalry is gone. That of sophisters, economists, and calculators has succeeded; and the glory of Europe is extinguished forever.*
>
> – Edmund Burke (1729–1797)

> *For the study of political economy you need no special knowledge, no extensive library, no costly laboratory. You do not even need text-books nor teachers, if you will but think for yourselves.*
>
> – Henry George (1839–1897)

Like any course in macroeconomics, this one has several different aims. The course should help you: learn how models work; understand the models macroeconomists typically use; understand how shocks affect the economy; think about current and historical events in a disciplined way; and acquire research skills.

A character in Anne Tyler's *The Accidental Tourist* says that he works in quality control in a pencil factory, but "It's not nearly as interesting as it sounds." I hope that you find the opposite is true here.

(a) Questions as Moments.

What are the types of facts which we seek to explain in macroeconomics? Macroeconomics is an applied field, and we'll refer to the types of evidence with which particular models are designed to be consistent and to other facts which could be used to test them.

Some of the questions in macroeconomics are facts about levels of variables or about distribution: Why is the rate of unemployment so different from zero? Why can Depressions occur? Why can a low level of development persist in most countries? Why is most unemployment accounted for by many spells experienced by some rather than a few spells by all (which would constitute better insurance)? Why are shocks to output associated with changes in hours worked by the employed rather than with new employment? Why do some developed countries grow so much faster than others?

Other questions concern the dynamics of growth and cycles in aggregate measures within individual countries. What explains changes over time in the rate of growth? What accounts for business cycles, *i.e.* the persistence of shocks or deviations from trend growth? How do we explain the volatility of some measures, such as nominal exchange rates?

In the past, much economics proceeded as if growth and cycle theory were separate disciplines. To some extent, the course reflects this still, but ultimately these two are inseparable statistically and theoretically. A good model should account for both growth and cycles – though this is very hard to do - and conversely, growth and cycle evidence should be used to test ideas. A particular fact is consistent with many theories, and additional facts (as well as other, theoretical criteria) can be used to eliminate some possible explanations.

Now, many of these questions can be summarized as *statistical moments*: means (conditional), variances, covariances, and autocovariances. Here are some examples:

Aggregate consumption tends to be smooth over the business cycle (smoother than output), investment more volatile and highly procyclical, net exports countercyclical, and the real wage acyclical. The money supply and nominal income move together to some extent, as do output and unemployment, and the nominal exchange rate and the price level. We can formally describe these in terms of moments: consumption is highly autocorrelated, investment has a high variance, the covariance between the real wage and output is roughly zero.

Particular theories sometimes are associated with particular correlations: money and nominal income with the quantity theory of money, the nominal exchange rate and the price level with PPP, and so on.

Remember that moments are simply averages of some property in historical data. For example, consider the correlation between the government budget deficit and the general level of interest rates. If, on average, interest rates are high relative to their mean when deficits are high relative to their mean, then the correlation is positive. You can think of a goal of macroeconomic modelling as being able to explain this sort of correlation (the scale, not just the sign) and reproduce it in a model. You might also think that this is a modest goal, and you would be right. After all, shouldn't we try to explain the actual time series paths of macroeconomic variables, and forecast them? We should, but first we should ensure that our model(s) are consistent with the more general evidence summarized in moments.

We can sensibly calculate moments only for random variables that are *stationary*. Loosely-speaking, a variable is stationary if it does not have a trend. For a trending variable, such as the level of GDP, we can calculate a sample mean, but that value would not mean much and would not help us learn about other time periods. Hence we may detrend (or calcualte growth rates) before calculating moments.

Let us next look at some specific facts in greater detail. First consider growth and fluctuations in output and employment. At the most basic level, the objective of macroeconomic theory is to account for growth and fluctuations in national output, measured conveniently by GNP. From 1959-1996 GNP grew, on average, by about 3-4 percent per year in North America. [Growth rates here refer to log differences.] The fluctuations are also evident from a time series graph, although over periods this long they seem fairly small relative to the steady growth. At shorter intervals the fluctuations take on greater importance. We see in the Table 1 for the U.S. and in Table 2 for Canada that the standard deviation of quarterly growth rates is 4-5 percent. Certainly on a quarter to quarter basis, fluctuations in growth rates are large relative to the mean.

TABLE 1

Properties of Growth Rates: U.S. Data

Variable	Mean	Std Deviation
GNP	3.07	3.83
Consumption (nondur, serv)	3.09	1.94
Fixed investment	3.67	9.83
Employment	2.05	2.85
Output per worker	0.88	2.83
Price Level (cpi)	4.00	3.45
Money Stock (M2)	6.85	6.42
Velocity	0.76	7.00

Statistics refer to log differences, and have been multiplied by 400 to make them annual percentages. The sample period is 1959:2- 1996:3, with the exception of employment and prices, which are 1947:2-1996:3.

TABLE 2

Properties of Growth Rates: Canadian Data

Variable	Mean	Std Deviation
GNP	3.85	5.01
Consumption (nondur, serv)	3.54	3.66
Fixed investment	4.65	11.37
Employment	1.67	2.47
Output per worker	0.92	2.51
Price Level (cpi)	4.47	4.06
Money Stock (M2)	9.96	5.19
Velocity	-1.17	4.66

Statistics refer to log differences, and have been multiplied by 400 to make them annual percentages. The sample period is 1947:2-1996:3, with the exception of money and velocity, which are 1968:2-1996:3, and employment and productivity which are 1976:2-1996:3.

The same features appear in the components of GNP. We see for consumption and investment that the growth has been comparable. The fluctuations, however, show up much more strongly in investment than in consumption or GNP as a whole. One of the apparent features of fluctuations, then, is that production of durable goods, here measured by fixed investment (plant and equipment, housing) is relatively volatile. Another robust feature of short term fluctuations is that consumption is a very smooth series (excluding durables, which behave like investment). But for nondurables and services the standard deviation of growth rates is about half that of GNP as a whole, and about a fifth that of investment.

One of the things you might ask yourself is whether the growth and fluctuations in GNP reflect changes over time in the amount of labour used to produce the output or changes in labour productivity. One can compare movements in the 'labour input', measured by total hours worked in the economy, with those in real GNP. Note that changes in hours can be due to changes in the population size, in participation (perhaps related to age and gender), in the employment rate, and in average hours per employee. Anyway, one of the striking features here is the steady growth in output per hour worked. Labour productivity, by this measure, has increased by an average of 0.9 percent per year in the US over the period. This, too, is a common feature of macroeconomic data, as is a similar trend in real payments to labour. The source of this productivity growth may be the most important issue in macroeconomics. We shall see the implications of this growth in section 4, and discuss explanations for the productivity growth slowdown.

Movements in the labour input also have been important in fluctuations. As you doubtless know, the amount of labour varies quite a bit over the business cycle. By our measurement, almost as much as GNP. Nevertheless, we can detect cyclical movements in average labour productivity. There obviously are also movements in the unemployment rate. From Table 2 we know that total hours have increased and because average weekly hours have not been trending up that must reflect an increase in employment. But the unemployment rate has also risen over this historical period so participation in the labour force must have risen even faster.

So far the evidence we have examined has concerned real quantities. A second set of macroeconomic variables concerns the price level, the dollar price of a basket of goods. For obvious reasons, the price level tends to be associated with the stock of money. We see that these two variables also exhibit strong trends over the postwar period, with the money stock somewhat steeper. We can think of this in terms of the relation

$$m + v = p + y,$$

where m is the money stock, v is velocity, p is the price level, and y is national income (GNP), all in logarithms. This equation, with the additional assumption that v is constant, is often referred to as the quantity theory, but for now let us simply regard it as the definition of velocity.

As an accounting matter, we can decompose the movements in p into movements in m, y, and v. Over the postwar period, it appears that movements in m have been the most important. In North America since 1945 $m - y$ has the same trend as p, so that v has no trend to speak of, though it is certainly variable. (This is for M2.) But notice also that annual rates of growth of m and p differ sharply.

The examples given so far have been *unconditional moments.* Sometimes economic questions can be described best with *conditional moments.* This distinction should be familiar to you from introductory statistics, where you distinguished between the unconditional and conditional expectation of a random variable. This difference simply reflects the fact that a forecast or expectation depends on what information is available. The same distinction can be drawn for other moments, such as variances, as we'll see in the exercises.

Many questions in open-economy macroeconomics also can be described using moments. In connection with questions about the international economy we shall study two-country equilibrium models in section 2 and a model of volatile exchange rates in section 3. See if you can explain why any of the following generalizations is surprising, or challenging to explain:

The correlation between international interest differentials and subsequent exchange rate changes is low or even negative, a result called the forward premium anomaly. The variation in real exchange rates is nearly as great as the variation in nominal exchange rates, and real exchange rates are highly autocorrelated. The correlation of consumption growth across countries often is lower than the correlation in income growth. The correlation between savings and investment is high, either in a cross-section of countries or in a time series for an individual country.

Economists don't yet have an internally consistent framework which also is consistent with all these moments, but agreeing on some of the key questions is a start.

(b) Types of Models

Macroeconomic models may be classified in a number of ways. One dimension concerns the genetics of the agent(s) in the model. Ironically much of macro is now done as if there were a single agent in the economy. This is sometimes called the *representative agent* assumption or model. Under certain conditions (homothetic preferences so that there are no wealth effects) we can aggregate individual behaviour and model the economy as if there were one agent, or do international economics as if there were one agent in each country. (I'm not counting the government here - there is now a lot of theory applied to the relation between the private and public sectors). In certain applications we'll work with models in which the representative agent lives forever (is infinitely-lived); in others, for simplicity, we'll assume that the economy ends in finite time, which might be after two periods. A tractable alternative to the representative-agent model is the structure known as *overlapping generations*, in which there are at least two types of agents (young and old) alive at each moment. Obviously certain types of trades and policies (such as bequests or social security plans) can be studied in this type of model but not in a model with a single type of agent. These types of structures are introduced in sections 2 and 4 of the course respectively.

Models also might be classified according to whether markets clear or not. Market-clearing models are sometimes called Walrasian or equilibrium models and are associated with classical economics. In these models, wages and prices adjust instantly to equate supply and demand, and the economy is always at its full employment level. Non-market-clearing models are sometimes called disequilibrium models and tend to be identified with Keynesianism. Here wages and prices do not adjust instantly to equate demand and supply and so trade occurs at non-Walrasian prices.

A third way to classify models involves policy issues and the potential role of the government in stabilisation (or ultimately in increasing welfare). In an undergraduate course in macroeconomics, you have probably come across schools of thought – such as

Keynesianism or monetarism – and the associated views of macroeconomic policy. While these terms (Keynesianism and monetarism) may retain some meanings they are not at all precise. And most debates now cannot be summed up with these terms. For these reasons I wish these isms were wasms. More recently, references have been made to so-called new classical macroeconomics. This term was used in the late 1970s and early 1980s to refer to classical models (*i.e.* with market clearing) to which random disturbances were added to explain business cycle fluctuations and in which agents have rational expectations. In simple new classical models there is little role for government – only the unexpected component of monetary policy changes has a real effect and that effect is destabilizing. For new classical economics business cycles are not caused by market rigidities such as sticky prices. Instead, they are viewed as equilibrium phenomena, arising from the interaction of exogenous shocks to technology, say, and the optimal decisions of agents. Whether macroeconomics is the right place for the application of competitive, general equilibrium is a good question. Hicks argued that macro was the right place to use micro, since individual differences and specific deviations from competitive assumptions wash out in aggregates. One strong argument for competitive models is that they are relatively easy to work with.

A parallel name is new Keynesian macroeconomics. Again the term 'new' means that agents have rational expectations. Again there are random shocks to the economy. Markets do not clear so that there may be a greater role for government policy. For example, suppose that firms and unions set fixed nominal wages in a long-term contract. Before the contract expires the government can effect output as follows: if it adds to aggregate demand with expansionary policy the price level will rise. Nominal wages are fixed so real wages fall and labour supply will tend to change so that output will change. Recently, Keynesians have tended to focus more on the goods market than the labour market, and to examine reasons why goods prices might be sticky. Establishing microfoundations for these rigidities is important but not easy.

A fourth dimension along which models may be classified concerns their treatment of information. I refer to assumptions about how people form expectations, how much information they have, and whether there is diversity of beliefs across agents. Most macroeconomic models embody the idea of rational expectations, which we'll formalize in sections 2 and 3 of the course. This has nothing to do with views about the efficacy of macroeconomic policy. The idea behind this model of expectations is that, since economists often assume that firms maximize profits and consumers maximize utility, they might also assume that agents act rationally when forming expectations about the future. Agents can be modelled as having rational expectations whether or not markets clear continuously.

Notwithstanding the various different approaches to modelling there has been progress in macroeconomics. Current models tend to derive decision rules from (i) optimization problems, which makes them compatible with much of microeconomics. Most models are of the (ii) general equilibrium variety, so that all interdependencies are taken into account. The sophistication with which (iii) empirical evidence is used has also increased. Macroeconomics is the empirical application of dynamic, stochastic, general equilibrium models.

(c) Solving and Identifying Models

In order to answer an economic question with a model, we must first solve the model, then study its predictions. What does it mean to solve a model?

For our initial purposes a model consists of endogenous, or explained (or dependent) variables and exogenous, or forcing (or independent) variables. These are related through a set of equations which include:

- accounting identities such as those in the national accounts
- behavioural relationships, e.g. the consumption function, describing agents' strategies or rules of behaviour
- institutional rules such as tax schedules
- technological constraints such as production functions

What is exogenous and endogenous depends on what the purpose of the model is; there is no grand unifying model. Ultimately perhaps only weather (and now not even that) is exogenous; in other cases foreign variables are exogenous; in the short run the capital stock can be taken as given or exogenous, and so on.

Solving the model simply means solving this set of equations, figuring out how the endogenous variables depend on the exogenous variables. Here is a simple example, in a static, model that is deterministic (*i.e.* contains no random elements or uncertainty):

Suppose we have this model:

Example 1

$$y = c + i$$
$$c = \beta y$$

in which y and c are endogenous. Sometimes the model is called the *structural form* and the solution is called the *reduced form*. We want the reduced form in order to forecast, in order to learn about the parameters, and in order to do comparative statics or econometrics. The latter two activities make sense in the reduced form since we can sensibly speak of shocks to or changes in exogenous variables. When there is much interdependence in the things we are thinking about then logically we need the reduced form to avoid mistakes.

In this example the reduced form is

$$y = \frac{i}{1-\beta}$$
$$c = \frac{\beta i}{1-\beta}$$

or we could write $c = \beta y$ since the reduced form is recursive. Note that there is only one shock or exogenous variable *viz.* i. Therefore c and y move together, which fits the facts. If we had one shock or exogenous variable (particularly if unobservable) for each endogenous variable in a linear model then we might not have much of a model, since we couldn't ever find facts which would be inconsistent with it.

Comparative statics refers to seeing how a small change in an exogenous variable affects an endogenous variable. In example 1 above, $dy/di = 1/(1-\beta)$. This is called the multiplier in this income-expenditure model. Clearly the effect of changes in i on y depends on the value of β. We can differentiate again, this time with respect to the parameters, to see this effect:

$$\frac{d(dy/di)}{d\beta} = (1-\beta)^{-2} > 0$$

so that the larger is β the larger is the multiplier. This is called *sensitivity analysis*, seeing how sensitive the predictions of the model are to the assumed parameter values. In larger models, sensitivity analysis may be done by computer.

For a model to be useful, its parameters (β in this example) must be identified. *Identification* refers to learning the values of the parameters. To check on the identification, visualize the system as involving the unknown parameter β. Could you solve this system to write β as a function of the variables we observe? The system in example 1 is identified; from the reduced form we can find the value of the single parameter, β. In fact, it is over-identified. If the model is correct then the same β should appear in each equation. Over-identification is a good thing, for it provides a test of the model. If the model is a useful guide, then the same β should appear in each equation.

In the example, it was simple to solve the model by substitution. For larger systems we may use linear algebra or a numerical equation solver, such as Maple. What if the original structure (consisting of accounting identities and behavioural rules) is nonlinear? In that case, we'll almost always need to use numerical methods. However, we may be able to do comparative statics, as the next example shows.

Example 2

$$y = f(c, i)$$
$$c = g(y)$$

where f and g are some nonlinear functions. This example doesn't make sense economically, but bear with me. Even if there is no analytical or closed-form solution, total differentiation gives a linear system which we can solve.

$$dy = f_c dc + f_i di$$
$$dc = g_y dy$$

where $f_c = \partial f(c, i)/\partial c$ and so on. Now

$$dy/di = \frac{f_i}{1 - f_c g_y}$$

which generalizes the previous example. Notice a very important point: even though the original system was nonlinear, once we totally differentiate the system of differentials is linear and so can be solved readily. And we could use linear algebra to manipulate more complicated (linear) totally differentiated systems. Here is a specific example for those unfamiliar with partial differentiation:

Example 3

$$y = c^{\alpha} i^{\beta}$$
$$c = exp(\gamma y)$$

so $f_c = \alpha c^{\alpha-1} i^{\beta}$, $f_i = \beta c^{\alpha} i^{\beta-1}$, $g_y = \gamma exp(\gamma y)$. Notice that, say, dy/di depends on the levels of i, c, and y, unlike the situation in the linear structure. We might suspect that dy/di is larger in recessions than in booms, for example. Here is a further, specific example:

Example 4

$$y = ln(c) + ln(i)$$

Can we find dy/di at $i = 0$? No.

So far these examples are static and deterministic. We use models with these properties in intermediate macroeconomics, and they can be quite revealing. In section 2 we shall turn to models with uncertainty *i.e.* to stochastic as opposed to deterministic models.

In stochastic models the exogenous variables are random variables. The easiest way to write these down is to add an error term to the deterministic functions above. This error term is also called a shock. It may be temporary or permanent. The shocks feed through the economic model give rise to stochastic models of the endogenous variables. In this sense, the economic model may contain propagation mechanisms. As an example, suppose that there is a decision to invest in physical capital. This is long-lived; suppose there is a big negative shock to capital productivity – an outbreak of rust or a computer virus. Then the current capital stock will be lower than otherwise and output in the following period will be lower than otherwise – there is persistence in the endogenous variable, output, even if the shock occurs only once.

We make models stochastic for several reasons. One is that economic agents face pervasive uncertainty and it is therefore realistic to use models with this feature whenever possible. Another is that our tools for testing economic theories typically are based on probabilistic statements (this is econometrics), which provide a measure of closeness between the predictions of a model and the actual, historical data. That is, we also regard history as a draw of a sequence from an urn – our economic models or theories are just urns or data generation processes and we seek to know how likely (with what probability) they could have generated the data which we observe.

Shocks in an economic model have mean zero, otherwise they would be partly systematic. Often they are characterized as independent of the exogenous variables, too. Usually they are simply added to exogenous variables and can be treated like additional exogenous variables that aren't observed directly. Here is an example:

Example 5

$$
\begin{aligned}
y &= c + i \\
c &= \beta y \\
i &\sim iin(0, \sigma^2)
\end{aligned}
$$

Here i is a shock with mean zero (otherwise its mean could be put in a constant term). In this example, use of the solution gives:

$$
\begin{aligned}
y &\sim iin\Big(0, \frac{\sigma^2}{(1-\beta)^2}\Big) \\
c &\sim iin\Big(0, \frac{\beta^2\sigma^2}{(1-\beta)^2}\Big)
\end{aligned}
$$

Is the variance of c less than the variance of y? What is the correlation between c and y? This example illustrates a general point. The statistical properties of the endogenous variables depend on (i) those of the exogenous variables or shocks, and (ii) the properties of the model. In this example there is no propagation, and hence no persistence, because the shock is *iid*. What would happen if the shock followed a first-order autocorrelation?

If we knew the parameter values, then the predictions could be compared to historical moments, such as those listed earlier in Tables 1 and 2. Also the parameters β and σ^2 could be identified from the variances of consumption and income. These conditions then can allow econometric estimation of the parameters. We shall discuss method-of-moments estimation, of which this is an example, in section 5.

In dynamic models the exogenous variables will be sequences. We can imagine various patterns of shocks; temporary, permanent changes and so on. If these are impulses, then the economic model is a propagation mechanism or filter through which they pass to give rise to sequences of values for the endogenous variables. It is the properties of these sequences (*e.g.* their means, variances, and autocorrelations) which we can compare to history.

Solving a dynamic model means finding the entire time paths of variables following some shock or change in an exogenous variable (be sure you understand why we cannot speak of a shock to an endogenous variable). In most of our models the dynamics will be described by linear, difference or differential equations. In dynamic models we may count time discretely or continuously depending on a number of issues, and study difference or differential equations accordingly. In continuous time we usually write $y(t)$, while in discrete time we usually write y_t.

Often, the variables in a dynamic model will tend to some long-run values, which we refer to as the *steady state*. For example, a variable might tend to gradually return to its steady-state value after a shock. Of course we'll be interested in the transition path along the way as well as the steady state itself. Sometimes the solution will involve continuing growth, rather than constant values. An economy that tends to return to a path with a constant growth rate for several variables is said to have a *balanced growth path*.

Most dynamic models are also stochastic and involve expectations, and we shall discuss their manipulation in section 3 of the course. In some cases we ignore randomness; this usually is called the assumption of perfect foresight.

So far, we have talked as if models were simply given and we have studied how to deduce their implications. But of course the hard part is writing down a model in the first place, and in practice people evaluate models not only by the degree of correspondence between their implications and reality. So where do the hypothesised behavioural relationships come from? In neoclassical economics they generally arise as the solutions to optimization problems attributed to agents.

In most cases the optimization problems ascribed are dynamic *e.g.* as in the permanent income hypothesis. Their mathematical solution gives rise to first-order (necessary) conditions which will be difference equations (if the problem is in discrete time). Perhaps the most general development in macroeconomics recently has been the tendency to make these optimization problems explicit. Since dynamic optimization is unfamiliar, in the first part course we shall simply state first-order conditions when we come across dynamic problems or else derive them from two-period special cases. But later, in section 5, you'll be introduced to *dynamic programming*, a powerful way to solve dynamic, stochastic model economies.

(d) Eight Tools

As a final step in our introduction, here is a list of eight helpful tools that we'll use in the course that follows. The description of the tools here is quite brief, but they will become more familiar with practice.

1. *Calculator.* A hand calculator is one the the most useful tools in macroeconomics. Working out numerical examples often helps one understand and apply theory, as I hope you'll see during this course.

2. *Log approximation.* If x is a small number, then

$$\ln(1 + x) \approx x.$$

Try checking the accuracy of this rule-of-thumb using your calculator.

3. *Growth rates.* In continuous time, the growth rate of a variable x is

$$\dot{x} \equiv \frac{dx/dt}{x} = \frac{d\ln x}{dt}.$$

The growth rate of a ratio is the difference between the two underlying growth rates:

$$\dot{\left(\frac{x}{y}\right)} = \dot{x} - \dot{y}.$$

The growth rate of a product is the sum:

$$\dot{(xy)} = \dot{x} + \dot{y}.$$

You can easily prove these two rules using calculus.

In discrete time, the growth rate is:

$$\frac{x_t - x_{t-1}}{x_{t-1}}.$$

(Sometimes x_t/x_{t-1} is called the gross growth rate.) For small growth rates, we can approximate the growth rate as

$$\frac{x_t - x_{t-1}}{x_{t-1}} \approx \ln x_t - \ln x_{t-1} = \Delta \ln x_t,$$

where Δ denotes a difference. This is simply an example of rule **1**. Because this is an approximation, the sum and difference rules for growth rates of products or ratios also are only approximate in discrete time.

4. *Discounting and power series.* We'll often come across an infinite sum in the form of a power series, and need to know if it converges. If $|x| < 1$ then

$$\sum_{i=0}^{\infty} x^i = 1 + x + x^2 + ... = \frac{1}{1-x}.$$

For example, imagine calculating the present value of a stream of constant payments denoted b, using an interest rate r:

$$b + \frac{b}{1+r} + \frac{b}{(1+r)^2} + ... = \frac{1+r}{r}b,$$

where we simply use our formula with $x = 1/(1+r)$. What is the present value if the stream begins next year instead of this year?

Very rarely, we may need to find a finite sum, like this: $1 + x + x^2$. A helpful trick is to write this as:

$$\begin{aligned} 1 + x + x^2 &= \sum_{i=0}^{\infty} x^i - (x^3 + x^4 + x^5 + ...) \\ &= \frac{1}{1-x} - x^3(1 + x + x^2 + ...) \\ &= \frac{1-x^3}{1-x} \end{aligned}$$

Discounting in continuous time looks a bit different. The present value of a constant stream b is:

$$\int_0^{\infty} exp(-ri)b\, di = -\frac{b}{r} exp(-ri)\Big]_0^{\infty} = \frac{b}{r}.$$

where now r is the instantaneous interest rate.

5. *Moments of functions of random variables.* Suppose that x is a random variable, with mean μ and variance σ^2. Something else, say y, depends on x. Suppose the relationship is linear:

$$y = a + bx,$$

then the mean of y is $a + b\mu$ and the variance of y is $b^2\sigma^2$. Scaling something affects both its mean and its standard deviation by the same factor, b.

If the relationship is nonlinear, say some function $y = f(x)$, then we cannot find the moments of y without knowing the probability density function of x. It is helpful to remember something that is *not* true. When f is nonlinear,

$$E(y) \equiv E[f(x)] \neq f[E(x)] \equiv f(\mu).$$

Instead,

$$E[f(x)] > f[E(x)] \quad \text{if} \quad f'' > 0,$$

and

$$E[f(x)] < f[E(x)] \quad \text{if} \quad f'' < 0,$$

which is called *Jensen's Inequality.*

If we are given the complete density of x then we can work out the density of y and hence find its moments precisely. We'll see both discrete and continuous examples of this later in the notes.

6. *Lag operator.* When we keep track of macroeconomic variables in discrete time, observations at different dates can be denoted with the lag operator, L. It is defined this way:

$$Lx_t \equiv x_{t-1},$$

so that

$$L^i x_t \equiv x_{t-i}.$$

The characteristic equation of a difference equation is simply a polynomial in the lag operator.

7. *Solving difference equations forwards or backwards.* Suppose that a variable x_t follows a stochastic difference equation (*i.e.* a difference equation with an error term added):

$$x_t = \rho x_{t-1} + \epsilon_t,$$

where the shock or error term is an independently and identically distributed random variable. Lagging this and substituting for x_{t-1} gives:

$$x_t = \rho(\rho x_{t-2} + \epsilon_{t-1}) + \epsilon_t.$$

This is the first step in solving the difference equation backwards. If you repeat this, you will see that x_t can be written as depending on a series of past shocks and an initial condition.

At other times we'll solve difference equatios forward, again by repeated substitution. Usually it will be obvious which direction to choose.

8. *Covariance decomposition.* Suppose that x and y are random variables and that we need to study the forecast or expectation of their product. Then we use the covariance decomposition, which is:

$$E(xy) = E(x)(y) + cov(x, y).$$

To prove the decomposition, simply use the definition of the covariance. This rule reminds us that the expectation of the product is *not* equal to the product of expectations, unless the covariance between the two variables is zero.

Further Reading

If any of the mathematics used in this section is unfamiliar to you, then you should refer to a textbook in mathematical economics. A good example is Alpha Chiang's *Fundamental Methods of Mathematical Economics* (1984).

Many of the time series methods used in macroeconomics are best learned from textbooks in econometrics. A book that is at exactly the right level for this course is Walter Enders's *Applied Econometric Time Series* (1995).

For an informal, lucid discussion of the requirements and aims of business-cycle theory, I recommend Robert Lucas's essay , "Understanding business cycles," 215-239 in his book *Studies in Business Cycle Theory* (1981).

Exercises

1. Consider the following structure:

$$\begin{aligned}
y_d &= c + i + g \\
i &= \delta + \gamma r \\
c &= \alpha + \beta y \\
m - p &= \phi r \\
y_s &= \theta + \epsilon p
\end{aligned}$$

where y, i, c, g, r, m, and p are variables and greek letters denote constants.

(a) What is the most standard division of this list into exogenous and endogenous variables? Why?

(b) Find the reduced-form equations for y and p.

(c) Is the marginal propensity to consume identified from the reduced-form equation for y? *i.e.* Could policy-makers learn the value of β by varying m and g and observing y?

2. [difficult] Consider the following model of aggregate consumption:

$$c_t = E_t(y_{t+1}^{\beta}), \quad \beta \epsilon (0, 1)$$

where c_t is the level of consumption this period, and y_{t+1} is the level of income next period. y_t is a random variable.

Professor N. Aive proposes to test this model by following these steps:

(i) collect data on y and c, calculate the average of each series, denoted $\overline{c}$ and $\overline{y}$.

(ii) find an estimate of β, denoted $\hat{\beta}$, as the value that solves the following nonlinear equation:

$$\overline{c} = \overline{y}^{\beta}$$

(iii) test the theory by comparing the sequence $\{c_t\}$ with the sequence $\{y_t^{\hat{\beta}}\}$.

Professor J. Ensen argues that the latter sequence will systematically underpredict the values of the former sequence. Which scholar is correct and why?

3. Suppose an economy has the following structure:

$$\begin{aligned}
y &= c + i + g \\
c &= \beta y \\
i &= -\delta r \\
g &= -\gamma y \\
m &= -\mu r \\
m &= \eta (r - r^*)
\end{aligned}$$

where greek letters are positive parameters, and r^* is an exogenous target interest rate. All variables are real and the price level is fixed.

(a) List the endogenous variables.

(b) Find the reduced form of the model (not in recursive form).

(c) Suppose that the government announces its target r^* each period. Show how one could identify β from the reduced-form coefficients.

(d) Are the parameters over, just, or under-identified?

4. The behaviour of endogenous variables depends on the properties of shocks as well as of the structure. Consider the multiplier-accelerator model of aggregate demand, which dates from the 1930s:

$$\begin{aligned} y_t &= c_t + i_t + g_t \\ c_t &= \beta y_{t-1} \quad 0 < \beta < 1 \\ i_t &= \iota(y_t - y_{t-1}) \quad\ 0 < \iota \end{aligned}$$

The series $\{g_t\}$ is exogenous.

(a) Find the first-order difference equation describing the evolution of y_t. *i.e.* relate y_t to predetermined and exogenous variables.

(b) For the remainder of this exercise, assume that $\iota = 0.5$ and $y_0 = 1$. Suppose that $\beta = 0.8$. Consider the sequence for $\{g_t : t = 1, 2, 3, 4, 5\}$ of $\{2, -2, 2, -2, 2\}$. Find $\{y_t : t = 1, 2, 3, 4, 5\}$.

(c) Now suppose that $\beta = 0.9$ while ι and $\{g_t\}$ are unchanged. Is the income sequence now more or less persistent than was the case in part (b)?

(d) Now suppose that $\beta = 0.8$ again but that $\{g_t : t = 1, 2, 3, 4, 5\}$ is given by $\{2, 0, -2, 0, 2\}$. Again find $\{y_t : t = 1, 2, 3, 4, 5\}$. Is the income sequence now more or less persistent than was the case in part (b)?

5. Here is a simple model of real money and real income:

$$\begin{aligned} m &= \delta y + \gamma r \\ y &= \alpha m \end{aligned}$$

with r exogenous.

(a) Which (if any) of the three parameters is identified?

(b) By differentiating the reduced-form coefficients with respect to the underlying parameters establish the truth or falsity of the following claim: the lower the income elasticity of the demand for money (δ) the greater the absolute size of the response of income to a change in the interest rate.

(c) Suppose that $r \sim iid(0, \sigma^2)$. Find the means and variance of y and m and their covariance. (One also could show the dependence of these moments on the underlying parameters but that is not necessary here.)

6. [difficult] This exercise addresses some issues in identification. Suppose that an endogenous variable y is related to two exogenous shocks:

$$y_t = \alpha \epsilon_t + \beta \eta_t,$$

where α and β are positive parameters. Suppose that $\epsilon_t \sim iin(\mu_\epsilon, \sigma^2_\epsilon)$ and $\eta_t \sim iin(\mu_\eta, \sigma^2_\eta)$ and that their covariance is zero. But ϵ and η are unobservable.

(a) Find the mean and variance of y.

(b) To learn which shock is more important we would like to know the parameter values: α, β, μ_ϵ, σ_ϵ, μ_η, σ_η. Are they identified?

(c) Suppose that $\alpha = \beta = 1$ and that we observe another variable $x_t = \eta_t$. Can we identify the remaining parameters?

(d) [difficult] Now suppose that y is not observed. With the information in part (c), what is $E(y_t|x_t)$?

Answer

(a) $E(y) = \alpha\mu_\epsilon + \beta\mu_\eta$; $\text{var}(y) = \alpha^2\sigma^2_\epsilon + \beta^2\sigma^2_\eta$.

(b) None is. We have two equations in six unknowns. By normality the first two moments contain all the information.

(c) We can learn μ_η and σ_η from the moments of x. Then if $\alpha = \beta = 1$, we have: $E(y) - E(x) = \mu_\epsilon$ and $\text{var}(y) - \text{var}(x) = \sigma^2_\epsilon$, because the covariance is zero.

Incidentally, if α were not unity we could learn it from the ratio of these two moments.

(d) $E[y_t|x_t] = E[\epsilon_t + \eta_t|\eta_t] = \eta_t + \mu_\epsilon$

7. This question outlines a simple, mathematical example of a propagation mechanism. Suppose that an exogenous shock ϵ_t affects an endogenous variable y_t as follows:

$$y_t = \lambda y_{t-1} + \epsilon_t$$

and that $\epsilon_t \sim iin(\mu, \sigma^2)$. Assume that λ is a positive fraction.

(a) By repeated substitution, write y_t in terms of current and past shocks only.

(b) Find the mean and variance of y.

Answer

(a) $y_t = \epsilon_t + \lambda\epsilon_{t-1} + \lambda^2\epsilon_{t-2} + \dots$

(b) $E(y) = \mu(1 + \lambda + \lambda^2 + \dots) = \mu/(1-\lambda)$ $\quad \text{var}(y) = \sigma^2(1 + \lambda^2 + \lambda^4 + \dots) = \sigma^2/(1-\lambda^2)$

where we use the independence of the various ϵ's. Notice that the moments of y depend both on the moments of the shock and on the propagation mechanism.

8. Suppose that a shock ϵ_t is normally distributed with mean zero and variance σ^2. It also is independently distributed over time, so it cannot be predicted based on past observations.

A second variable behaves this way:

$$x_t = \alpha + \rho x_{t-1} + \epsilon_t,$$

with $x(0) = 0$.

(a) Find the conditional mean and variance of x_{t+1} given x_t.

(b) By solving the difference equation backwards, find the unconditional mean and variance of x_t. Does the starting value, x_0 have a long-term effect?

(c) Find the autocorrelation function for any lag.

(d) Graph the impulse response function which describes the response of x_t, x_{t+1}, x_{t+2}, ... to a shock $\epsilon_t = \sigma$.

(e) Using your backwards solution in part (b), explain why the value of ρ characterizes how persistent the x-series is. How persistent are shocks when $\rho = 0$ and when $\rho = 1$?

(f) Is x stationary (*i.e.* does it have stable moments) for any value of ρ?

(g) On a computer, simulate a series of ϵ_t and hence construct and graph a realization of x_t, for several different values of ρ.