Queen's University
School of Graduate Studies and Research
Department of Economics

Economics 850 **Fall, 2023**

Professor James MacKinnon

Midterm Examination

October 24, 2023. Time Limit: 80 minutes

Please answer all 3 questions. The marks for each question are shown in square brackets. All answers should be as short as it is reasonably possible to make them.

1. Consider the linear regression model

$$y_i = \beta_1 + \beta_2 x_i + u_i, \quad i = 1, \ldots, N.$$

where the x_i are independent draws from the $\chi^2(1)$ distribution, and the u_i are independent draws from a normal distribution with mean 0 and unknown variance.

a) Is the OLS estimate of β_2 biased or unbiased in this case? Explain very briefly.

b) Is the OLS estimate of β_2 consistent or inconsistent in this case? Explain very briefly.

c) It is well-known that the leverage of each observation in the sample is proportional to a number called h_i. How would you compute the h_i in this case? If you sorted the observations by the value of x_i, which observation would have the highest leverage?

d) Suppose there are 100 observations. The OLS estimate of β_2 using the full sample is $\hat{\beta}_2$. If you throw away the 10 observations with the 10 largest values of h_i, you now have a sample with 90 observations. It yields another OLS estimate, say $\check{\beta}_2$. How is the variance of $\check{\beta}_2$ related to the variance of $\hat{\beta}_2$? Only one of the following statements is true. Which one is it? Briefly explain why.

(i) $\mathrm{Var}(\check{\beta}_2) = \frac{10}{9}\,\mathrm{Var}(\hat{\beta}_2)$.

(ii) $\mathrm{Var}(\check{\beta}_2) > \frac{10}{9}\,\mathrm{Var}(\hat{\beta}_2)$.

(iii) $\mathrm{Var}(\check{\beta}_2) < \frac{10}{9}\,\mathrm{Var}(\hat{\beta}_2)$.

(ii) $\mathrm{Var}(\check{\beta}_2)$ may be larger or smaller than $\frac{10}{9}\,\mathrm{Var}(\hat{\beta}_2)$.

[30]

`Please turn over. The other two questions are on the back.`

2. Consider the linear regression model

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{u} = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_2\boldsymbol{\beta}_2 + \boldsymbol{u}, \quad \boldsymbol{u} \sim \text{IID}(\mathbf{0}, \sigma^2\mathbf{I}),$$

where the notation is standard. There are N observations and $k = k_1 + k_2$ regressors.

a) Derive the covariance matrix of the OLS estimator $\hat{\boldsymbol{\beta}}_2$ using the FWL Theorem.

b) How would you estimate this matrix in practice? How would you obtain an estimated standard error for $\hat{\beta}_2$? Would the expectation of this standard error equal the true standard deviation of $\hat{\beta}_2$?

c) Suppose the matrix $\boldsymbol{X}_1$ in the above regression is replaced by the matrix $\boldsymbol{X}_1\boldsymbol{A}$, where $\boldsymbol{A}$ is a square matrix with full rank. How will this affect the OLS estimate of $\boldsymbol{\beta}_2$? Explain very briefly.

d) Suppose the matrix $\boldsymbol{A}$ of part c) has rank $k_1 - r$ instead of rank k_1. Would replacing $\boldsymbol{X}_1$ by $\boldsymbol{X}_1\boldsymbol{A}$ affect the OLS estimate of $\boldsymbol{\beta}_2$ now? Explain very briefly.

e) OLS estimation of the model above yields a residual vector $\hat{\boldsymbol{u}}$. What do you know about the SSR $\hat{\boldsymbol{u}}^\top\hat{\boldsymbol{u}}$? How is the SSR related to the (unobservable) quantity $\boldsymbol{u}^\top\boldsymbol{u}$? What is the expectation of the SSR?

[35]

3. Consider the linear regression model

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{u} = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \beta_2\boldsymbol{x}_2 + \boldsymbol{u}, \quad \boldsymbol{u} \sim \text{IID}(\mathbf{0}, \sigma^2\mathbf{I}),$$

where the notation is standard. There are N observations and $k = k_1 + 1$ regressors. All the regressors are assumed to be exogenous.

a) Explain precisely how you would test the hypothesis that $\beta_2 = 1$ at the .01 level. Would the test you are proposing be exact for finite N?

b) If the null hypothesis that $\beta_2 = 1$ is true, what can you say about the distribution of the OLS estimator $\hat{\beta}_2$?

c) Suppose that, in fact, $\beta_2 = 0.75$. Now what can you say about the distribution of the OLS estimator $\hat{\beta}_2$?

d) Can you compute the power of the test of part a), or at least an asymptotic approximation to its power, under the assumption that $\beta_2 = 0.75$? If so, how?

e) Based on the procedure you proposed in part d), explain how to compute either an exact or an approximate power function for the test of part a) as a function of the true value of β_2. What would be (perhaps approximately) the lowest value of this function, and for what value of β_2 would it achieve this value?

[35]