Queen's University
School of Graduate Studies and Research
Department of Economics

Economics 850 **Fall, 2022**

Professor James MacKinnon

Midterm Examination

October 25, 2022. Time Limit: 80 minutes

Please answer all 3 questions. The marks for each question are shown in square brackets. All answers should be as short as it is reasonably possible to make them.

1. Suppose that you have obtained 100 observations $[y_i, x_i]$ on a regressand y and a regressor x. In your sample, x_i equals 0 for the first 80 observations and 1 for the remaining 20. The model you are concerned with is $y_i = \beta_1 + \beta_2 x_i + u_i$, where the disturbances u_i are assumed to be independent with mean 0.

a) According to this model, what is the expectation of y conditional on $x = 1$? What is the unconditional expectation of y if the probability that $x = 1$ is the same in the population as in the sample?

b) For this sample, some observations, the ones that provide more information about β_2, have higher leverage than the others. Which observations are the ones with relatively high leverage?

c) If the u_i are known to be distributed as $\text{NID}(0, \sigma^2)$, how would you test the hypothesis that $\beta_2 = 0$ at the .05 level? Would this be an exact test?

d) Would the test of part c) still be exact if the u_i actually followed the distribution just below, which you (the investigator) do not know?

$$u_i \sim \text{N}(0, 1) \text{ with probability } 0.9$$
$$u_i \sim \text{N}(0, 25) \text{ with probability } 0.1$$

If you used the test of part c) in this case and obtained a P value of 0.0392, do you think it would be safe to reject the null hypothesis at the .05 level?

[32 marks]

2. Consider the linear regression model

$$\boldsymbol{y} = \beta_1 \boldsymbol{x}_1 + \boldsymbol{X}_2 \boldsymbol{\beta}_2 + \boldsymbol{u}, \quad \boldsymbol{u} \sim \text{IID}(\boldsymbol{0}, \sigma^2 \mathbf{I}), \tag{1}$$

where $\boldsymbol{y}$, $\boldsymbol{x}_1$, and $\boldsymbol{u}$ are n–vectors, and $\boldsymbol{X}_2$ is an $n \times k_2$ matrix, with $k = k_2 + 1$. Assume that all regressors are exogenous.

Continued on next page ...

a) If you impose the restriction that $\beta_1 = 1$ and it is false, will the restricted OLS estimator $\tilde{\boldsymbol{\beta}}_2$ be biased or unbiased (in general)?

b) Briefly explain how to obtain $\tilde{\boldsymbol{\beta}}_2$ using a regression package. What will be $\boldsymbol{V}(\tilde{\boldsymbol{\beta}}_2)$, the actual covariance matrix of $\tilde{\boldsymbol{\beta}}_2$ if the restriction $\beta_1 = 1$ is true, and what will be $\hat{\boldsymbol{V}}(\tilde{\boldsymbol{\beta}}_2)$, your estimate of that matrix?

c) Let $\hat{\boldsymbol{\beta}}_2$ denote the OLS estimator when no restriction is imposed on β_1. What will be $\boldsymbol{V}(\hat{\boldsymbol{\beta}}_2)$ and $\hat{\boldsymbol{V}}(\hat{\boldsymbol{\beta}}_2)$? Why is $\tilde{\boldsymbol{\beta}}_2$, in general, more efficient than $\hat{\boldsymbol{\beta}}_2$ when the restriction is true.

d) Is there any special case in which $\tilde{\boldsymbol{\beta}}_2$ will *not* be more efficient than $\hat{\boldsymbol{\beta}}_2$ even though the restriction $\beta_1 = 1$ is true? Explain briefly.

[32 marks]

3. Consider the linear regression model $\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{u}$, where there are N observations and k regressors. Assume that the regressors are random, the disturbances are independently distributed with mean zero and the same finite variance, and the probability limit of $\boldsymbol{X}^\top\boldsymbol{X}/N$ is the positive definite matrix $\boldsymbol{S}_{\boldsymbol{X}^\top\boldsymbol{X}}$.

Under standard assumptions, each of the quantities below is of some stochastic order in the sample size. Thus each of them is $O_p(N^c)$ for some value of c which varies from case to case.

a) Precisely what is the value of c (or, equivalently, the value of N^c) for each of the following quantities?

   i. $\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0$, where $\hat{\boldsymbol{\beta}}$ is the OLS estimator and $\boldsymbol{\beta}_0$ is the true parameter vector.

   ii. $\boldsymbol{X}^\top\boldsymbol{u}$.

   iii. $\boldsymbol{u}^\top\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{u}$, where $\boldsymbol{M}_{\boldsymbol{X}} = \mathbf{I} - \boldsymbol{X}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}^\top$.

   iv. $\boldsymbol{u}^\top\boldsymbol{P}_{\boldsymbol{X}}\boldsymbol{u}$, where $\boldsymbol{P}_{\boldsymbol{X}} = \boldsymbol{X}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}^\top$.

   v. $\boldsymbol{y}^\top\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{y}/(N-k)$.

b) Now consider the case in which $\boldsymbol{y} = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_2\boldsymbol{\beta}_2 + \boldsymbol{u}$, where the notation should be obvious. Let $\boldsymbol{M}_1$ denote the matrix that projects orthogonally off $\mathcal{S}(\boldsymbol{X}_1)$. Precisely what is the value of c (or, equivalently, the value of N^c) for each of these two quantities?

   i. $\boldsymbol{y}^\top\boldsymbol{M}_1\boldsymbol{X}_2(\boldsymbol{X}_2^\top\boldsymbol{M}_1\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2^\top\boldsymbol{M}_1\boldsymbol{y}$ when $\boldsymbol{\beta}_2 = \mathbf{0}$.

   ii. $\boldsymbol{y}^\top\boldsymbol{M}_1\boldsymbol{X}_2(\boldsymbol{X}_2^\top\boldsymbol{M}_1\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2^\top\boldsymbol{M}_1\boldsymbol{y}$ when $\boldsymbol{\beta}_2 \neq \mathbf{0}$.

[36 marks: 22 for a); 14 for b)]