Queen's University
School of Graduate Studies and Research
Department of Economics

Economics 850 | **Econometrics I** | **Fall, 2019**

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Final Examination

December 9, 2019. Time Limit: 3 hours

Please answer any **four (4)** of the following **six (6)** questions. Each question has **four parts** and is worth 25% of the final mark. Before deciding which questions to answer, it would be a very good idea to read every question carefully.

Note: A table with some critical values of the χ^2 distribution appears at the end of the examination.

1. Consider the linear regression model

$$\boldsymbol{y} = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_2\boldsymbol{\beta}_2 + \boldsymbol{u}, \tag{1}$$

where there are n observations and k exogenous regressors, with k_1 regressors in the matrix $\boldsymbol{X}_1$ and k_2 regressors in the matrix $\boldsymbol{X}_2$. The elements u_i of $\boldsymbol{u}$ are assumed to be uncorrelated but to have different variances σ_i^2, which are unknown.

a) Explain how you would test the hypothesis that $\boldsymbol{\beta}_2 = \boldsymbol{0}$ using an asymptotic test. Write down the test statistic explicitly as a function of $\boldsymbol{y}$, $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, and functions of those things, using projection matrices to simplify the algebra. How is this statistic distributed asymptotically under the null hypothesis?

b) Explain how you would generate B bootstrap samples for a bootstrap test of the hypothesis $\boldsymbol{\beta}_2 = \boldsymbol{0}$ based on the test statistic of part a).

c) Let τ denote the actual test statistic from part a) and τ_b denote the bootstrap statistic for the b^{th} bootstrap sample from part b). Precisely how would you compute the bootstrap P value for the test of $\boldsymbol{\beta}_2 = \boldsymbol{0}$.

d) Suppose a student used $B = 100$ and obtained a bootstrap P value of 0.07 using the procedure of parts b) and c). The student concludes that the hypothesis that $\boldsymbol{\beta}_2 = \boldsymbol{0}$ cannot be rejected at the .05 level. Would you congratulate the student on a job well done? Why or why not?

2. Consider the nonlinear regression model

$$y_i = \beta_1 + \beta_2(x_i^\alpha + z_i^\alpha) + u_i, \quad \mathrm{E}(u_i \,|\, x_i, z_i) = 0, \tag{2}$$

where the disturbances u_i are assumed to be independent, but it is not assumed that they are homoskedastic.

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a) Explain how you could you test the hypothesis that $\alpha = 0.5$ without doing any nonlinear estimation at all.

b) Suppose that you estimate equation (2) by nonlinear least squares. If the program you are using is not capable of computing heteroskedasticity-robust standard errors for NLS estimates, but can do so for OLS estimates, explain how you would compute an asymptotically valid standard error for $\hat{\alpha}$ and how you would use it to construct a 95% confidence interval.

c) Suppose equation (2) is to be estimated using a sample of 2150 observations, of which 1110 come from Ontario and 1040 come from Quebec. You wish to test the null hypothesis that all the parameters are the same for the two provinces against the alternative that they are different. Explain how to do so using NLS just once (for the restricted model) along with one OLS regression. How is your test statistic distributed asymptotically under the null?

d) Now explain how to test the hypothesis of part c) using NLS just once (for an unrestricted model). Will your test statistic be numerically equal to the one from part c)? Will it have the same asymptotic distribution under the null? Explain briefly.

3. Consider the linear regression model

$$y_i = \beta_1 + \beta_2 x_{2i} + \beta_3 x_{3i} + u_i, \tag{3}$$

which is to be estimated using a sample of 187 observations. The regressors x_{2i} and x_{3i} are assumed to be exogenous. You are interested in the parameter $\gamma \equiv \beta_2/\beta_3$.

a) Explain how you would obtain an estimate $\hat{\gamma}$ and an asymptotically valid standard error $s(\hat{\gamma})$ analytically (i.e., without doing any simulations) under the assumption that the disturbances in (3) are independent but may display heteroskedasticity of unknown form.

b) Explain how you would obtain a bootstrap standard error $s^*(\hat{\gamma})$ under the assumptions of part a). How could you use that standard error to construct a confidence interval for γ? Would your interval be symmetric around $\hat{\gamma}$?

c) Explain how you would perform a bootstrap test of the hypothesis that $\gamma = 1.25$ under the assumptions of part a).

d) Explain how you would construct a studentized bootstrap confidence interval for γ under the assumptions of part a). Would your interval be symmetric around $\hat{\gamma}$? Why or why not?

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4. Suppose that you have samples of households drawn from ten different cities, where, for the i^{th} city, the sample is of size n_i. You believe that an appropriate model is

$$\boldsymbol{y}_i = \alpha_i \boldsymbol{\iota}_i + \boldsymbol{X}_i \boldsymbol{\beta} + \boldsymbol{u}_i, \quad \mathrm{E}(\boldsymbol{u}_i \boldsymbol{u}_i^\top) = \boldsymbol{\Omega}_i. \tag{4}$$

Here the $\boldsymbol{y}_i$ are $n_i \times 1$ vectors of observations on the dependent variables, the $\boldsymbol{\iota}_i$ are $n_i \times 1$ vectors of 1s, the $\boldsymbol{u}_i$ are $n_i \times 1$ vectors of disturbances, the $\boldsymbol{X}_i$ are $n_i \times k$ matrices of observations on exogenous variables, and the $\boldsymbol{\Omega}_i$ are positive definite covariance matrices of dimension $n_i \times n_i$.

a) Explain how you would estimate the parameters α_i, $i = 1, \ldots, 10$, and $\boldsymbol{\beta}$ jointly by ordinary least squares. Then explain how you would estimate the covariance matrix of $\hat{\boldsymbol{\beta}}$, the OLS estimate of $\boldsymbol{\beta}$.

b) Regression (4) involves $k+10$ regressors. Explain how you could obtain exactly the same estimates $\hat{\boldsymbol{\beta}}$ by running a regression with only k regressors. How would you estimate the covariance matrix of $\hat{\boldsymbol{\beta}}$ now?

c) Suppose you wish to test the restrictions that $\beta_1 = \beta_2 = \beta_3$, where these parameters are the first three elements of $\boldsymbol{\beta}$. Explain how you would compute a test statistic that asymptotically follows a $\chi^2(r)$ distribution. What is the value of r? If the test statistic were 6.253, would the null hypothesis be rejected at the .05 level using an asymptotic test? Would it be wise to rely on such a test in this case?

d) Briefly explain how you could instead perform a bootstrap test of the same null hypothesis. If $\hat{\tau} = 6.253$ is the actual test statistic and τ_b denotes the b^{th} bootstrap statistic out of 999 such statistics, precisely how would you calculate the bootstrap P value?

5. Consider the linear regression model

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{u} = \boldsymbol{Z}\boldsymbol{\beta}_1 + \boldsymbol{Y}\boldsymbol{\beta}_2 + \boldsymbol{u}. \tag{5}$$

Here $\boldsymbol{y}$ and $\boldsymbol{u}$ are $n \times 1$, $\boldsymbol{X} \equiv [\boldsymbol{Z}\ \boldsymbol{Y}]$ is $n \times k$, $\boldsymbol{Z}$ is $n \times k_1$, and $\boldsymbol{Y}$ is $n \times k_2$. The columns of $\boldsymbol{Z}$ are exogenous regressors, while the columns of $\boldsymbol{Y}$ are possibly endogenous. The k–vector of coefficients $\boldsymbol{\beta}$ is divided into two subvectors $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ of dimensions k_1 and k_2, respectively.

a) Suppose that, in addition to $\boldsymbol{Z}$, you have available an $n \times \ell$ matrix of instruments $\boldsymbol{W}$, with ℓ large enough so that $\boldsymbol{\beta}$ is either exactly identified or overidentified. What is the IV estimator $\hat{\boldsymbol{\beta}}$? This estimator can be obtained by minimizing a certain criterion function. What is that function?

b) How many overidentifying restrictions are there? Explain how you can test these overidentifying restrictions under the assumption that the elements of $\boldsymbol{u}$ are homoskedastic. What would you conclude if $n = 17{,}243$, $k_1 = 12$, $k_2 = 2$, $\ell = 5$, and the value of your test statistic were 15.24?

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c) Asymptotically, it must be true that

$$n^{1/2}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0) = \boldsymbol{A}\boldsymbol{d}, \tag{6}$$

where $\boldsymbol{A}$ is a $k \times k$ matrix and $\boldsymbol{d}$ is a $k \times 1$ vector. Rewrite equation (6) using the actual expressions for $\boldsymbol{A}$ and $\boldsymbol{d}$, making sure that they include the appropriate factors of n,

d) Suppose that the i^{th} element u_i of $\boldsymbol{u}$ has mean 0 and variance σ_i^2. Based on your answer to part c), you should be able to derive the asymptotic distribution of $n^{1/2}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)$. What is this distribution? Just give the result. Do not attempt to prove anything.

6. Consider the linear regression model

$$\boldsymbol{y} = \beta_1 \boldsymbol{\iota} + \beta_2 \boldsymbol{x} + \boldsymbol{u}, \quad \boldsymbol{u} \sim \text{NID}(\boldsymbol{0}, \sigma^2 \mathbf{I}), \tag{2}$$

where all vectors have n elements. Here $\boldsymbol{\iota}$ denotes a vector of 1s, and $\boldsymbol{x}$ denotes a vector of observations on an exogenous regressor.

a) Write down the t statistic for $\beta_2 = 1$ explicitly and explain how to make a regression package display it as part of the output from a linear regression. Precisely how is this statistic distributed under the null hypothesis?

b) How is the t statistic of part a) distributed under the fixed alternative that $\beta_2 = 1 + \delta$, for $\delta \neq 0$?

c) Using the result of part b), explain how to calculate the power of a one-tailed test that $\beta_2 \leq 1$ against the alternative that $\beta_2 > 1$ as a function of δ. How would power for $\delta = 0.3$ change if the sample size n were doubled? How would it change if σ were doubled?

d) Again using the result of part b), explain how to calculate the power of a two-tailed test that $\beta_2 = 1$ against the alternative that $\beta_2 \neq 1$ as a function of δ. When $\delta > 0$, will this test be more powerful than the test of part c), equally powerful, or less powerful? Explain carefully.

Table 1. Some Critical Values of the χ^2 Distribution

D.F. / Level	.10	.05	.025	.01
1	2.706	3.841	5.024	6.635
2	4.605	5.991	7.378	9.210
3	6.251	7.815	9.348	11.345
4	7.779	9.488	11.143	13.277
5	9.236	11.070	12.833	15.086
6	10.645	12.592	14.449	16.812