KEY

SELF-ASSESSMENT TEST

GRADUATE METHODS REVIEW

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I. Basic Notation

A. Next to the symbol in the left column put the letter of its meaning on the right. Put "?" for symbols that are not familiar to you. (An example is provided.)

a.	5	$\sum$	0. there exists
b.	11	$\prod$	1. equivalent (indifference) to
c.	4	$\forall$	2. union
d.	0	$\exists$	3. preferred to
e.	12	$\bigcap$	4. for all
f.	8	$\emptyset$	5. summation
g.	2	$\bigcup$	6. subset of
h.	9	$\lVert \cdot \rVert$	7. orthogonal to
i.	3	$\succeq$	8. empty set
j.	10	$\#$	9. norm
k.	7	$\perp$	10. cardinality of
l.	6	$\subseteq$	11. product
m.	1	$\sim$	12. intersection

B. Give a simple **and** correct use of each of the following operands. An example is provided.

a. $\sum$ $\quad \sum_{k=0}^{3} \frac{1}{2}^{k} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8}$

b. $\prod$ $\quad \prod_{k=0}^{3} \frac{1}{2}^{k} = 1\frac{1}{2}\frac{1}{4}\frac{1}{8} = \frac{1}{64}$

c. $\bigcup$ $\quad \{1,2,3\} \bigcup \{3,4\} = \{1,2,3,4\}$

d. $\bigcap$ $\quad \{1,2,3\} \bigcap \{3,4\} = \{3\}$

e. lim $\quad \lim_{x \to 0} \frac{1}{1+x} = 1$

f. $|\ \ |$ or det( ) $\quad \begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} = 2$

C. Give a simple **and** correct example of the following mathematical objects.

a. A lower-triangular matrix: $\begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix}$

b. An unbiased estimate of the population mean (based on a sample of size N)

$$\bar{X} \equiv \frac{1}{N} \sum_{i=1}^{N} X_i$$

c. A singular matrix: $\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}$

d. A non-monotonic function (a formula not a graph): $f(x) = x^2$

D. Translate this symbolic statement into an English sentence:

$$\exists\, U \in \Re \;:\; \forall x \in \Re,\; |f(x)| \leq U.$$

There exists a real number U such that for any number x, the absolute value of $f(x)$ is less than or equal to U.

What kind of function is $f(x)$? **BOUNDED**

E. Translate this English sentence into symbolic notation:

For every element of the set S there is a neighborhood around it that is fully contained in S.

$$\forall s \in S \exists \epsilon_s > 0 : x \in S \forall x : \|x - s\| < \epsilon_s.$$

What kind of set is S? **OPEN**

II. Definitions

F. Let $f(x) : \Re \rightarrow \Re$. Complete the definition of the derivative:

$$f'(x) \equiv \lim_{\epsilon \to 0} \frac{\mathbf{f(x+\epsilon) - f(x)}}{\epsilon}$$

G. For the following subsets of the real line (a-d), put a "$\surd$" for **all** properties that apply. Put "?" for properties you are not sure of.

	a. $[0,1]$	b. $(-\infty, 2.2)$	c. $[0,2] \bigcup (3,\infty)$	d. $\{1, 1/2, \ldots, 1/n, \ldots\}$
1. open	—	$\surd$	—	—
2. bounded	$\surd$	—	—	$\surd$
3. closed	$\surd$	—	—	—
4. connected	$\surd$	$\surd$	—	—
5. compact	$\surd$	—	—	—

H. For the following functions (a-d), put a "$\surd$" for **all** properties that apply. Put "?" for properties you are not sure of.

	a. $f(x) = 3$	b. $f(x) = x^2$	c. $f(x) = \lvert x \rvert$	d. $f(x) = e^{-x^2}$
1. bounded	$\surd$	—	—	$\surd$
2. continuous at $x = 0$	$\surd$	$\surd$	$\surd$	$\surd$
3. differentiable at $x = 0$	$\surd$	$\surd$	—	$\surd$
4. concave	$\surd$	—	—	—

I. Let X and Y be two random variables, with joint density $f(x,y)$. Then X and Y are said to be *independent* if

there exists functions $f_x(x)$ and $f_y(y)$ such that $f(x,y) = f_x(x)f_y(y)$ for all x and y.

J. Let X be a discrete random variable with density

$$f(x) = \begin{cases} 1/3 & \text{if } x = 0 \\ 1/2 & \text{if } x = 1 \\ 1/6 & \text{if } x = 3 \\ 0 & \text{otherwise} \end{cases}.$$

Compute the population mean and variance of X. (You can leave your answer in a form ready to be calculated, such as "$1 + 3.5 + (6 - 3.77)^2$".)

a. $E[X] =$ $0\frac{1}{3} + 1\frac{1}{2} + 3\frac{1}{6} = 1$.

b. $Var[X] =$ $(0-1)^2\frac{1}{3} + (1-1)^2\frac{1}{2} + (3-1)^2\frac{1}{6} = 1$.

K. Let Z denote the *standard normal random variable* (mean 0 and variance 1.0.

a. Sketch the density of Z: **SEE ANY STATS BOOK**

b. Fill in the missing elements $< d >$ and $< x >$ of the formula for the density function of Z (usually denoted either $f(z)$ or $\phi(z)$):

$$f(z) = \frac{1}{< d >} \exp\left\{ -\frac{1}{2} < x > \right\} \quad \textit{where} < d > = \sqrt{\mathbf{2}\pi} \;\textit{and}\; < x > = \mathbf{z^2}.$$

L. Let $Z_1, Z_2, Z_3, \ldots Z_{99}$ denote independent standard normal random variables (as many as needed). Provide formulas using the $Z's$ that follow these distributions:

a. the t with 10 degrees of freedom: $t = \frac{Z_1}{\sqrt{\sum_{n=2}^{11} / 10}}$.

b. the χ^2 distribution with 9 degrees of freedom: $\chi_9^2 = \sum_{n=1}^{9} Z_i^2$.

c. the F distribution with $15, 35$ degrees of freedom: $F_{15,35} = \frac{\frac{1}{15} \sum_{n=1}^{15} Z_n^2}{\frac{1}{35} \sum_{n=16}^{50} Z_n^2}$.

III. Functional Forms in Economics

M. Let $U(c) = \sum_{t=0}^{K} \beta^k E[u(c_t)]$ be a consumer's utility over a consumption path $c = (c_0, c_2, \ldots, c_K)$.

a. The consumer is *risk*

$$\begin{cases} \textit{averse} & \text{if} \quad u'' < 0 \text{ (more generally, strictly concave)} \\ \textit{neutral} & \text{if} \quad u'' = 0 \\ \textit{loving} & \text{if} \quad u'' > 0 \text{ (more generally, strictly convex)}. \end{cases}$$

b. β is called the **discount factor**.

c. If $K = \infty$, $|\beta| < 1$, and $c_t = \bar{c}$ for all t, then $U(c)$ simplifies to $u(\bar{c})/(1-\beta)$.

N. Write down the general form of each of these types of utility (or production) functions, defined over two goods, x_1 and x_2. (Sketch an indifference (or isoquant) curve for the first two.)

a. Cobb-Douglas:

$$u(x_1, x_2) = \mathbf{A}\mathbf{x_1}^{\alpha}\mathbf{x_2}^{\beta}$$

b. Leontief:

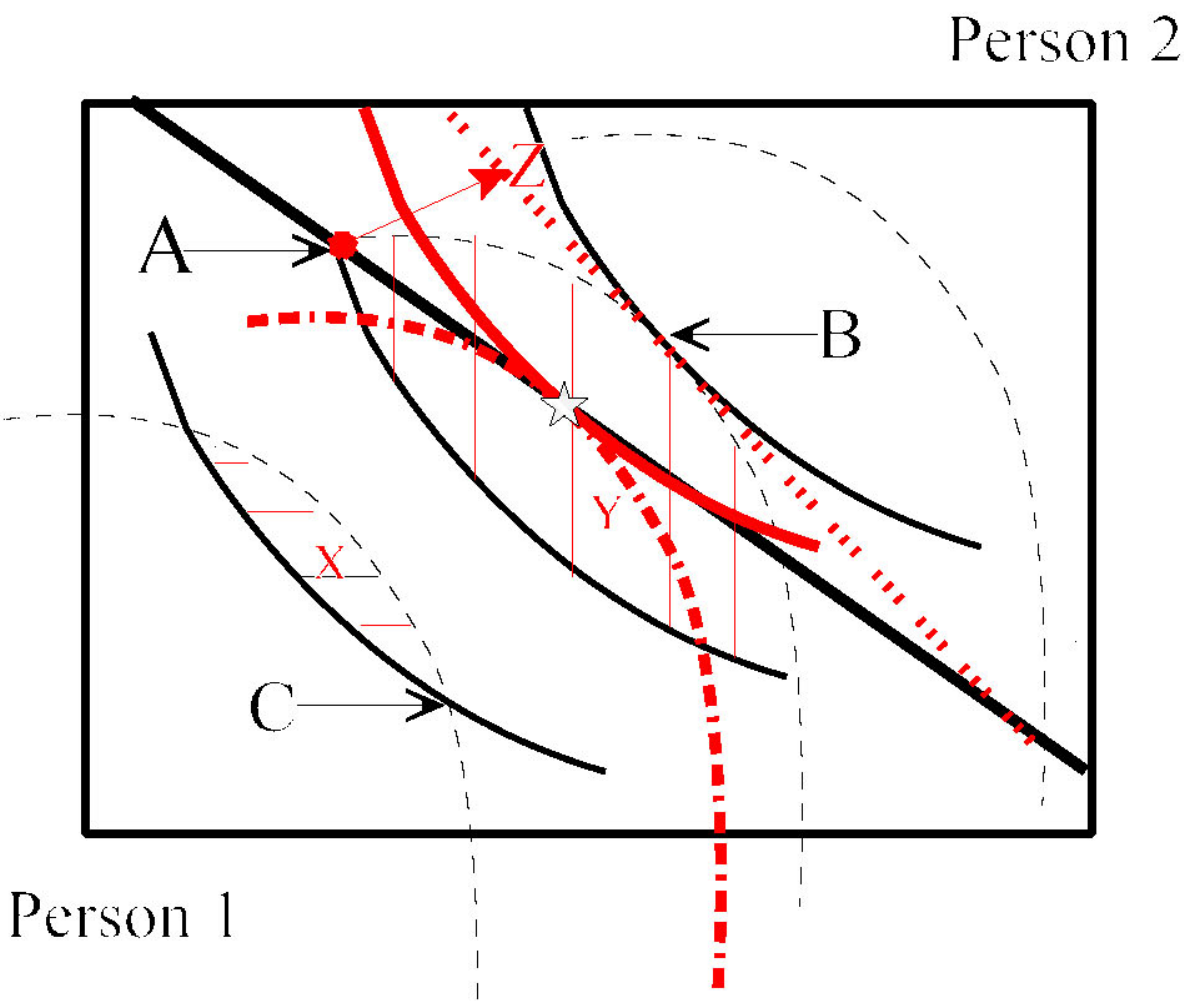
$$u(x_1, x_2) = \mathbf{A} \min\{\alpha \mathbf{x_1}, \beta \mathbf{x_2}\}$$

c. CES (Constant-elasticity-of-substitution):

$$u(x_1, x_2) = \mathbf{A}\left(\alpha \mathbf{x_1}^{\rho} + (\mathbf{1} - \alpha)\mathbf{x_2}^{\rho}\right)^{\mathbf{1}/\rho}$$

O. Apply the concepts below to the Edgeworth Box. (The solid curves represent Person 1's preferences; the dotted curves represent Person 2's preferences. Point: A is the *endowment*.)

- **a.** Shade the allocations that are *Pareto Superior* to C and label the area **X**.
- **b.** Shade the allocations in the *core* and label the area **Y**.
- **c.** List which of the allocations A, B, and C are on the *contract curve*:**B**
- **d.** Add an indifference curve for Person 1 and Person 2 that would illustrate a *competitive equilibrium* allocation on the price line through A.
- **e.** Illustrate a *lump sum re-allocation* that would result in B being a *competitive* allocation. Label the new endowment **Z**.



IV. Problem Solving

P. Consider the function:

$$f(x) = 1 + x - x^3/6.$$

a. Derive the first order necessary condition (FONC) for a point $x^\star$ to be a local optimal value of $f(x)$.

FONC: **if $x^\star$ is a local min/max, then $1 - [x^\star]^2/2 = 0$.**

b. Solve for all solutions to the FONC

$x^\star = \pm\sqrt{2}$

c. Derive the second order necessary condition for an optimal value

SONC: **if $x^\star$ is a local min [max], then $-2x^\star \geq [\leq]\ 0$.**

d. Conclude whether each of your solution(s) in b is a *maximum/minimum/inflection.*

1. $x^\star = \sqrt{2}$. This point satisfies FONC and SONC for a local maximum. It cannot be ruled out as a local maximum on this basis. However, since it also satisfies the second order SUFFICIENT condition for a maximum $(-2x < 0)$ it can be concluded it is a LOCAL MAXIMUM.

2. $x^\star = -\sqrt{2}$. This point satisfies FONC and SONC for a minimum. It cannot be ruled out as a local minimum. However, since it satisfies the second order SUFFICIENT condition $(-2x > 0)$ it can be concluded it is a LOCAL MINIMUM.

NB: no global optimal points exist. The point $x = 0$ is the one and only inflection point.