Numerical Methods for Economics and Econometrics

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I. Introduction by way of Example

I.1 Solving the basic optimal stopping problem

Examples

- ⋄ job search*
- ⋄ American options
- ⋄ retirement
- ⋄ patent renewal

Basic Assumptions

1. a simple *work* or *stay unemployed* decision

2. a decision to work is permanent

3. a known number of periods T to search

4. a constant value of continuing to search

5. a known and constant distribution of wage offers

I.1.a One Period (T) Decision

A person is unemployed coming into period T. The person values staying at home at some value α. The person has a wage offer in hand, w_T. The choice: stay unemployed (U) or work (W). The values of the two choices are:

$$V_T^U = \alpha$$

$$V_T^W(w_T) = w_T. \tag{1}$$

The optimal choice (or *optimal decision rule*) is: work if $V_T^W(w_T) \geq V_T^U$, or if $w_T \geq \alpha$. This defines the lowest wage offer, the *reservation wage*, that will be accepted by the person in period T:

$$w_T^\star = \alpha. \tag{2}$$

Given the optimal decision rule, we can determine the *indirect value* of of entering period T unemployed with the offer w_T in hand:

$$\begin{aligned} V_T(w_T) &= \max\{V_T^U, V_T^W(w_T)\} \\ &= \max\{\alpha, w_T\} \\ &= \max\{w_T^\star, w_T\} \end{aligned} \tag{3}$$

I.1.b Two Period (T and T-1) Decision

Enter T-1 unemployed, with wage offer w_{T-1} in hand. If the job offer is accepted, the job lasts today and tomorrow. Can either take the job or stay at home today and search again tomorrow. Tomorrow's wage offers are drawn from the distribution $F(w)$:

$$\begin{aligned} F(w) &= Prob(\text{wage offer} < w) \\ f(w) &= F'(w) = \text{density of wage offers} \end{aligned} \tag{4}$$

Income tomorrow is discounted by the rate β, $0 \leq \beta < 1$. An unemployed person makes a decision in period T-1 to accept the job offer or not to maximize discounted expected income.

The values of the parameters $(T, \alpha, \beta, F())$ define the *basic search model.*

Given the distribution of wage offers expected in time T, we can write down the expected income of arriving in period T unemployed:

$$\begin{aligned} EV_T &= \int_0^\infty V_T(w_T) f(w_T) dw_T \\ &= \alpha F(\alpha) + \int_\alpha^\infty w_T f(w_T) dw_T \\ &= w_T^\star F(w_T^\star) + \int_{w_T^\star}^\infty w_T f(w_T) dw_T \end{aligned} \tag{5}$$

The value taking of the two choices at T-1 are then

$$
\begin{aligned}
V^W_{T-1}(w_{T-1}) &= w_{T-1} + \beta w_{T-1} \\
V^U_{T-1} &= \alpha + \beta EV_T
\end{aligned}
\tag{6}
$$

The optimal decision: work if $V^W_{T-1}(w_{T-1}) \geq V^U_{T-1}$. This defines the lowest wage offer acceptable at time $T-1$:

$$
w^\star_{T-1} = \frac{\alpha + \beta EV_T}{1+\beta} \geq w^\star_T \tag{7}
$$

This in turn determines the indirect value of

$$
\begin{aligned}
V_{T-1}(w_{T-1}) &= \max\{V^U_{T-1}, V^W_{T-1}(w_{T-1})\} \\
&= \max\{\alpha + \beta EV_T, (1+\beta)w_{T-1}\} \\
&= (1+\beta)\max\{w^\star_{T-1}, w_{T-1}\}
\end{aligned}
\tag{8}
$$

I.1.c General Finite Period $(1, 2, \ldots, T)$ Decision

The person enters an arbitrary period t unemployed with wage offer w_t in hand. If the job offer is accepted, the job lasts through period T. The person can either take the job or stay at home today and search again tomorrow. Tomorrow's wage offers are drawn from the distribution $F(w)$. If we have solved backwards from $t = T$, then the expected (indirect) value of entering period t+1 unemployed is a value EV_{t+1}. This in turn determines the value of the two choices at period t:

$$
\begin{aligned}
V^W_t(w_t) &= \sum_{k=t}^{T} \beta^{k-t} w_t = w_t \frac{1-\beta^{T-t+1}}{1-\beta} \\
V^U_t &= \alpha + \beta EV_{t+1}
\end{aligned}
\tag{9}
$$

The optimal decision: work if $V^W_t(w_t) \geq V^U_t$. This defines the lowest wage offer acceptable at time t:

$$
w^\star_t = (\alpha + \beta EV_{t+1}) \frac{1-\beta}{1-\beta^{T-t+1}} \geq w^\star_{t+1} \tag{10}
$$

The indirect value of a wage offer at time t:

$$\begin{aligned} V_t(w_t) &= \max\{V_t^U, V_t^W(w_t)\} \\ &= \max\{\alpha + \beta EV_{t+1}, V_t^W(w_t)\} \\ &= \frac{1-\beta^{T-t+1}}{1-\beta}\max\{w_t^\star, w_t\} \end{aligned} \tag{11}$$

From this we can compute the value of entering period t unemployed:

$$\begin{aligned} EV_t &= \int_0^\infty V_t(w_t)f(w_t)dw_t \\ &= (\alpha + \beta EV_{t+1})F(w_t^\star) + \int_{w_t^\star}^\infty w_t f(w_t)dw_t \\ &= \frac{1-\beta^{T-t+1}}{1-\beta}\left(w_t^\star F(w_t^\star) + \int_{w_t^\star}^\infty w_t f(w_t)dw_t\right) \end{aligned} \tag{12}$$

This allows backward recursion to continue for $t-1, t-2, \ldots, 1$. Period T fits into this genernal formula if we define $EV_{T+1} = 0$. With that definition the equations $w_t^\star, V_t, EV_t$ can be used for any period t within a finite decision horizon.

I.1.d Infinite Horizon Problem (T=∞)

If we let T go to infinity then the decision horizon disappears. Today is just like tomorrow, in the sense that tomorrow is no more closer to T than today. The person enters an arbitrary period ('today') with wage offer w in hand. If the job offer is accepted, the job lasts forever. The person can either take the job or stay at home today and search again next period ('tomorrow'). Tomorrow's wage offers are drawn from the distribution $F(w)$. The value of taking a job offer in any period is the present discounted value of the stream of wages:

$$V^W(w) = \sum_{k=t}^\infty \beta^{k-t}w_t = \lim_{T\to\infty} w_t\frac{1-\beta^{T-t+1}}{1-\beta} = \frac{w_t}{1-\beta} \quad \text{for} 0 \le \beta < 1 \tag{13}$$

The value of not taking an offer is the expected value of staying at home and then entering tomorrow unemployed:

$$V^U = \alpha + \beta EV \tag{14}$$

The term EV has yet to be defined. In the finite horizon case, it could be determined before the optimal decision rule today. But today and tomorrow are the same, so in the infinite horizon problem the optimal decision rule and the indirect value of unemployment must be determined *simultaneously*. Whatever value EV is, the optimal decision rule is still simple: work if $V^W(w) \geq V^U$. This defines the lowest wage offer acceptable at any time:

$$w^\star = (\alpha + \beta EV)(1 - \beta) \tag{15}$$

Given a reservation wage $w^\star$ we could determine the value of unemployment:

$$\begin{aligned} V(w) &= \max\{V^U, V^W(w)\} \\ &= \max\{\alpha + \beta EV, V^W(w)\} \\ &= \frac{1}{1-\beta}\max\{w^\star, w\} \end{aligned} \tag{16}$$

And given the reservation wage we can determine the expected value of unemployment:

$$\begin{aligned} EV &= \int_0^\infty V(w)f(w)dw \\ &= \frac{1}{1-\beta}(w^\star F(w^\star) + \int_{w^\star}^\infty wf(w)dw) \end{aligned} \tag{17}$$

The equations for EV and $w^\star$ form a *system of simultaneous equations* that determine optimal decisions. One way to solve this system of equations is to begin with the period T problem and continue backward until $w^\star_t$ converges. This will work because it is straightforward to show that for $0 \leq \beta < 1$

$$\lim_{T\rightarrow\infty} w^\star_1 = w^\star. \tag{18}$$

That is, as the horizon disappears the reservation wage converges to the infinite horizon reservation wage.

I.2 Properties of the Basic Job Search Model

I.2.a Response of $w^\star$ to α and β

The basic job search model we have set up has a small number of parameters that determine the optimal process of job search. These parameters are the value of a period spent unemployed (α), the discount factor (β), and the distribution of wage offers ($f(w)$).

It is possible to use integration by parts to write an implicit equation for $w^\star$ that does not depend on EV as well. That is, the reservation wage in the infinite horizon model satisfies:

$$w^\star - \alpha = \beta\big(E(w) - \alpha\big) + \beta \int_0^{w^\star} F(z)dz. \tag{19}$$

(This is tedious to show and you are not required to know this equation.) Given this result, we can it to see how the reservation wage responds to the parmaeters of the model.

$$\frac{dw^\star}{d\alpha} - \frac{d\alpha}{d\alpha} = -\beta\frac{d\alpha}{d\alpha} + \beta F(w)\frac{dw^\star}{d\alpha} \tag{20}$$

This is an expression for the *total derivative* of the reservation wage with respect to a change in the parameter α. It is analogous to a *comparative static* exercise in micro theory, in the sense that it measures the response of the optimal decision rules to an exogenous parameter. Solve for the total derivative:

$$\frac{dw^\star}{d\alpha} = \frac{1-\beta}{1-\beta F(w^\star)} \in \{0,1\} \tag{21}$$

That is, the reservation wage goes up with the value of unemployment but not dollar-for-dollar. One way to think of this result is the following: an increase in α makes the person richer. Some of this increase in wealth is consumed in longer unemployment spells, but not all of it. If a job is rejected in order to stay unemployed and enjoy the higher value of α, then it may take several periods to get another acceptable offer. We know that $F(w^\star)$ is the probability of rejecting the next job offer as well.

As an exercise, derive $\frac{dw^\star}{d\beta}$.

I.3 Exponential Offer Distribution

Let

$$f(w) = \lambda e^{-\lambda w}, \quad w > 0 \tag{22}$$

which implies:

$$F(w) = 1 - e^{-\lambda w}, \quad w > 0$$

$$\int_x^\infty wf(w)dw = (1 - F(x))\,[x + \frac{1}{\lambda}]. \tag{23}$$

So $\Theta = (\alpha, \beta, \lambda)$ defines the basic, infinite-horizon, exponential-offer-distribution, search model. For this distribution,

$$EV = V^U(1 - e^{-\lambda w^\star}) + \frac{1}{1-\beta}\int_{w^\star}^{\infty} w\lambda e^{-\lambda w} dw \tag{24}$$

We can get a system of two equations that defines the model:

$$EV = \frac{w^\star}{1-\beta}\left(1 - e^{-\lambda w^\star}\right) + \frac{1}{1-\beta}\left[w^\star + \frac{1}{\lambda}\right] e^{-\lambda w^\star} \tag{25}$$

$$w^\star = (1-\beta)\left[\alpha + \beta EV\right] \tag{26}$$

As an exercise, derive $\frac{dw^\star}{d\lambda}$. It may simplify things to re-write the model as $\gamma = 1/\lambda$ and solve for $\frac{dw^\star}{d\gamma}$ first. Further, find the expressions for $\frac{dw^\star}{d\alpha}$ and $\frac{dw^\star}{d\beta}$ specific to this problem. If we put the total derivatives in a vector, we have:

$$\frac{dw^\star}{d\Theta} = \begin{pmatrix} \frac{dw^\star}{d\alpha} & \frac{dw^\star}{d\beta} & \frac{dw^\star}{d\lambda} \end{pmatrix} \tag{27}$$

II. The Search Model and Likelihood Estimation

II.1 The Search Model Can Produce a Linear Regression

The basic statistical model for earnings paid to individuals is the linear regression model in log-linear form:

$$\ln W = x\beta_x + u \tag{1}$$

Here x is a row vector of observed individual characteristics that are expected to influence earnings; β_x is a column vector of unknown parameters; and the scalar u is the difference between the expected log earnings offer $E[\ln W|x\beta_x] = x\beta_x$ and the actual value. It captures unobserved influences on a person's earnings. With $u \sim N(0, \sigma^2)$, the linear regression model with normally distributed errors is defined by $\Theta^r = (x, \beta_x, \sigma^2)$.

How does (1) relate to the infinite horizon search model (25)-(26) with parameters $\Theta = (\infty, \alpha, \beta, F())$? First, (1) introduces exogenous explanatory variables, x. If we set $\alpha = \beta = 0$, then the reservation wage becomes 0, and the searcher accepts any job offer. If we further set $lnw \sim N(x\beta_x, \sigma^2)$ we see that the regression model is a special case of the job search model, in which log wage offers are normally distributed with a constant variance σ^2 and an individual specific mean $x\beta_x$. That is, $\Theta^r(x, \beta_x, \sigma^2) = \Theta(\infty, 0, 0, N(x\beta_x, \sigma^2)) \equiv \Theta^\star$.

II.2 But It's Unlikely

The OLS estimate of $\hat{\beta}_x$ is $(X'X)^{-1}X'Y$, where X is the matrix of individual observations x and Y is the vector of observations on $\ln W$. Under the classical (Gauss-Markov) assumptions, $E[\hat{\beta}_x] = \beta$ and $Var[\hat{\beta}_x] = \sigma^2(X'X)^{-1}$. Suppose that one estimates (1) using OLS but the data were generated not by $\Theta^\star$ but by the more general model $\Theta = (\infty, \alpha, 0, N(x\beta_x, \sigma^2))$? That is, people still have no foresight but they do have an alternative to working with a value grater than 0.

What would the data look like? Let's start with the case that we get to observe x for a random sample, but that we don't observe earnings for people who rejected the wage offer.

Then all observed earnings would be greater than α. The residual u in (1) would no longer be have mean 0 but rather

$$E[u|x] = \sigma E[z|z > (\alpha - x\beta_x)/\sigma], \tag{2}$$

where $z \sim N(0,1)$. It follows that the error term would be correlated with x. That violates the key assumption of the Gauss-Markov Theorem. Thus, OLS estimates of (1) applied to the static search model ($\beta = 0$) results in biased estimates of β_x:

$$E[\hat{\beta}_x|\alpha > 0] \neq \beta_x.$$

This is an example of a *selection bias* in OLS estimates of a regression equation.

II.3 Instead it Produces a Likelihood Function

II.3.a Probit for Working

Under the normality assumption it turns out that

$$E[u|x] = \sigma E[z|z > d] = \sigma \frac{\phi(d)}{1 - \Phi(d)}, \tag{3}$$

where ϕ and Φ are the standard normal density and distribution function, respectively. Further, under search model the chances that a person chooses to work is $1 - F(\alpha) = 1 - \sigma\Phi(\frac{\alpha - x\beta_x}{\sigma})$. We see that the argument is exactly d.

Let's begin again with a model that says that the probability a person is employed is $1-\Phi(-x\gamma_x)$ and not employed $\Phi(-x\gamma_x)$. Code a new variable m to equal 1 for people working and $m = 0$ for those not working (created from the earnings data: $m = W > 0$). Then the probability of a given observation is

$$Prob(m|x, \gamma_x) = [1 - \Phi(-x\gamma_x)]^m [\Phi(-x\gamma_x)]^{1-m}. \tag{4}$$

The likelihood of the observation is this probability given the data:

$$L(\gamma_x|m, x) = Prob(m|x, \gamma_x) \tag{5}$$

The log-likelihood of a random sample of searchers:

$$\ln L(\gamma_x) = \sum_{i=1}^{N} \ln L(\gamma_x|m_i, x_i). \tag{6}$$

This function is globally concave in γ_x. The parameter vector that maximizes the sample likelihood are called *maximum likelihood estimates*:

$$\hat{\gamma}_x^{MLE} \equiv \arg\max_{\gamma_x} \ln L(\gamma_x). \tag{7}$$

MLE estimates of γ_x are consistent. The search model says that $\gamma_x = \beta_x/sigma$ except for the constant term, which would be $\gamma_x[1] = (\beta_x[1] - \alpha)/\sigma$. The parameter σ of the search model is not separately identified from the likelihood function for m alone.

We can generalize. Suppose that rather than having a common opportunity cost α, it is

$$\alpha = z\beta_z$$

where z is a vector of observed characteristics of the person (which may or may not overlap with x). Now the search model is $\Theta = (\infty, z\beta_z, 0, N(x\beta_x, \sigma^2))$. The working proability becomes $z\beta_z - x\beta_x$ and the *reduced-form* probability for m is

$$Prob(m|x, z, \beta_x, \beta_z) = \left[1 - \Phi\left(z\beta_z - x\beta_x\right)\right]^m \left[\Phi\left(z\beta_z - x\beta_x\right)\right]^{1-m}. \tag{8}$$

Based on this probability, we can't distinguish the homo- and heterogeneous opportunity cost search models, except by sorting the demographic variables into x and z *a priori*.

II.3.b Aside: The Heckman-Lee Two-Step Procedure

Given consistent estimates of the model's probability of working, we could correct the earnings regression for selection bias. First, compute an auxillary random variable

$$\hat{\lambda}([zx]) = \frac{\phi\left(z\hat{\beta}_z - x\hat{\beta}_x\right)}{1 - \Phi(\left(z\hat{\beta}_z - x\hat{\beta}_x\right))} \tag{9}$$

Then run the second-stage regress

$$\ln W = x\beta_x^\star + \gamma_\lambda \hat{\lambda}([zx]) + u_\lambda. \tag{10}$$

The OLS estimate of $\beta^\star$ is a consistent estimate of β.

The canonical form for the selection model is

$$\begin{aligned}
\text{observation:} Q &= (y, m, x_1, x_2) \\
y^\star &= x_1\beta_1 + \epsilon_1 && (11) \\
m^\star &= x_2\beta_2 + \epsilon_2 && (12) \\
m &= m^\star > 0 && (13) \\
y &= y^\star m && (14) \\
\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} &\sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2 & \rho\sigma \\ \rho\sigma & 1 \end{pmatrix}\right) && (15)
\end{aligned}$$

When compared to the search model, it is important to note that $m^\star$ is the difference between the offered wage and the reservation wage and its mean and variance have been normalized to 0 and 1, respectively. The full parameter vector is

$$\theta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \sigma \\ \rho \end{pmatrix}. \tag{16}$$

A truncated observation $(m = y = 0)$ occurs with probability $f(0,0) = P(m^\star < 0) = \Phi(-x_2\beta_2)$. The joint probability of a selected observation can be written $f(y,1) = f_y(y)f_m(1|y)$ where

$$\begin{aligned}
f_y(y) &= \frac{1}{\sigma}\phi((y - x_1\beta_1)/\sigma) \\
f_m(1|y) &= \int_{-x_2\beta_2}^{\infty} f(\epsilon_2|y)d\epsilon_2 \\
&= 1 - \Phi\left(\frac{-x_2\beta_2 - \rho(y - x_1\beta_1)/\sigma}{\sqrt{1-\rho^2}}\right)
\end{aligned}$$

The full log-likelihood of an observation given estimates $\hat{\theta}$ is

$$\begin{aligned}
\ln L(\hat{\theta}|Q) =& (1-m)\ln\Phi(-x_2\hat{\beta}_2) \\
&+ m\left[-\ln\hat{\sigma} - \frac{(y - x_1\hat{\beta}_1)^2}{2\hat{\sigma}^2} + \ln\left(1 - \Phi\left(\frac{-x_2\hat{\beta}_2 - \hat{\rho}(y - x_1\hat{\beta}_1)/\sigma}{\sqrt{1-\hat{\rho}^2}}\right)\right)\right]
\end{aligned} \tag{17}$$

II.3.c The Unrestricted Search Model

What happens when $\beta \neq 0$? Now the reservation wage $w^\star$ is influenced by both opportunity costs and the wage offer distribution. Estimating the reduced-form probit requires iteration. Since we are not assuming that offers follow the log-normal distribution the iteration includes the evaluation of an improper integral in (17). It is important to note that the searcher cares about the level of wages, so $f(w) \neq \phi(w)$ in (17). Instead, $f(w)$ is the density of the log-normal distribution.

If we ignore an important technical point, the likelihood function for a single observation is:

$$Prob(m, \ln W; \Theta) = \left[\Phi\left((w^\star(\Theta) - \beta_x x)/\sigma\right)\right]^{1-m} \left[\phi\left((\ln W - \beta_x x)/\sigma\right)/\sigma\right]^{m}. \tag{18}$$

The demographic variables x and z do not enter this expression symmetrically. The z variables only shift the reservation wage (the lower-bound) of observed data, whereas variables in x also shifts the distribution of accepted earnings above the reservation wage.

Discuss technical point ....

III. Integration and the Curse of Dimensionality

Notation: $x = (x_1, x_2, \ldots, x_n)$. Then $x_{\sim k} \equiv (x_{k+1}, x_{k+2}, \ldots, x_n)$, i.e., sub-vector selection and $z \sim v \equiv (z, v)$, i.e., row concatenation.

III.1 Numerical Integration

See the lucid discussion in Judd, 7.1-7.2

III.2 Integration in Higher Dimensions

You wish to evaluate numerically

$$I(l, u) \equiv \int_{l_m}^{u_m} \int_{l_m-1}^{u_{m-1}} \cdots \int_{l_1}^{u_1} f(x_1, x_2, \ldots, x_m) dx_1 dx_2 \ldots dx_m.$$

where l and u are $m \times 1$ vectors containing the limits to the integration in the corresponding dimensions of the function $f(x)$. A multi-dimensional integral can be written as a sequence of recursive one-dimensional integrals:

$$I^1(x_{\sim 1}, l, u) \equiv \int_{l_1}^{u_1} f(x_1, x_{\sim 1}) dx_1.$$

$$I^{k+1}(x_{\sim k+1}, l, u) \equiv \int_{l_{k+1}}^{u_{k+1}} I^k(x_{k+1} \sim x_{\sim k+1}) dx_{k+1}.$$

So that $I(l, u) = I^m(\emptyset, l, u)$. The cost of calculating a recursive problem for a given level of precision ϵ is exponential in m. Each time another dimension is added the same amount number of evaluation points nm must be added to maintain one-dimensional precisions. But for each of these points $n * n * n \ldots = n^{m-1}$ points in the recursion are added.

III.3 Monte Carlo Integration

Monte Carlo integration (8.2) breaks the curse of dimensionality in an average sense. That is, the precision (expressed as variance) of the Monte Carlo approximation to $I(l, u)$ is of order $1/\sqrt{N}$, where N is the number of replications, irrespective of dimension m. However, Monte Carlo integration is not smooth in the limits of integration l and u. Thus, for functions that require continuity (such as likelihoods) in which $I(l,u)$ enter Monte Carlo is not a solution.

III.4 Smooth Recursive Monte Carlo Integration

This problem can be resolved whenever the joint function $f(x)$ can be decomposed into a recursive sequence of functions that are easy to compute. In the case of a joint density, this means writing

$$f(x_1, x_2, \ldots, x_m) = f_1(x_1|x_{\sim 1})f_2(x_2|x_{\sim 2})\ldots f(x_m)$$

as the recursive product of conditional densities. In the case of jointly normal random variables the conditional densities are indeed easy to compute (they are just normal densities). So we can rewrite $I(l,u)$ as

$$\int_{l_k}^{u_k}\left[\int_{l_{k-1}}^{u_{k-1}}\left[\ldots\left[\int_{l_1}^{u_1} f_1(x_1|x_{\sim 1})dx_1\right] f_2(x_2|x_{\sim 2})dx_2\ldots\right] f(x_m)dx_m\right]$$

Notice

$$h_1(x_{\sim 1}) \equiv \int_{l_1}^{u_1} f_1(x_1|x_{\sim 1})dx_1 = F_1(u_1|x_{\sim 1}) - F_1(l_1|x_{\sim 1})$$

So given a random draw $\tilde{x}_2 \sim F_2(x_2|x_{\sim 2}, l_2 \le x_2 \le u_2)$, we can approximate the innermost two integrals with

$$h_1(\tilde{x}_2 \sim x_{\sim 2})\int_{l_2}^{u_2} f_3(x_2|x_{\sim 3})dx_2 = h_1(\tilde{x}_2 \sim x_{\sim 2})\left[F_2(u_2|x_{\sim 3}) - F_2(l_2|x_{\sim 3})\right] = h_1 h_2(x_{\sim 2}).$$

Proceeding recursively, we get a product

$$h = \prod_{k=1}^{m} h_k(\tilde{x}_{k-1} \sim x_{\sim k-1}).$$

It is not too difficult to show that $E[h] = I(l,u)$.