

Lecture notes 3

Economic growth: Solow model

1. Introduction

Solow's classic model is a superb piece of work, everything you could ask of a theory. It takes on the biggest questions—e.g., what determines standards of living, why some countries are rich and others poor. The argument is based on standard assumptions, yet it arrives at not-at-all obvious implications. It fits the facts well. So much so that Solow's model sets the framework for all serious empirical studies of growth and productivity.

Solow highlights technical change—i.e. productivity growth—as the key to long-run growth of per capita income and output. Accumulation of capital creates growth *in the long run* only to the extent that it embodies improved technology.

To develop the model, we start with the artificial situation of constant population and constant technology, and then, in steps, allow population to grow, and technology to improve.

2. The steady state

Production function

The aggregate production function is:

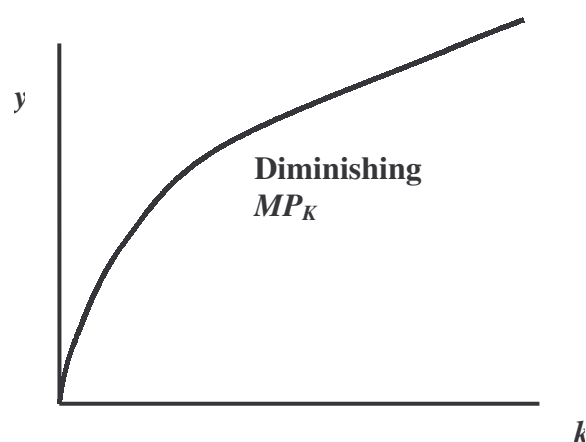
$$Y = F(K,L)$$

With constant returns to scale we can transform this into a function relating *output per worker* to *capital per worker*.

$$y = f(k)$$

where $y = Y/L$, and $k = K/L$.

Figure 3.1: Per worker production function



Accumulation of capital

The change in the capital stock per worker (known as *capital deepening*) is equal to per worker gross investment minus depreciation:

$$\Delta k = i - \delta k.$$

Ignore government for present purposes, so that investment is equal to private sector saving:

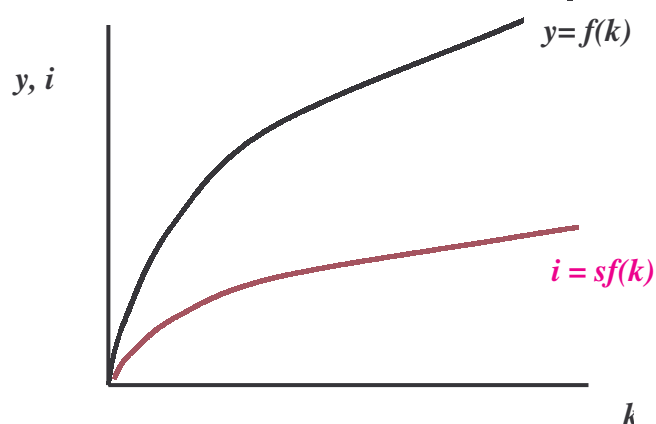
$$i = S/L = s Y/L = sy.$$

where s is the saving ratio (the MPS is for simplicity the same as the APS). This we can write in terms of the production function:

$$i = s f(k).$$

The proportional saving-income relationship implies that this investment function is like a scaled-down production function.

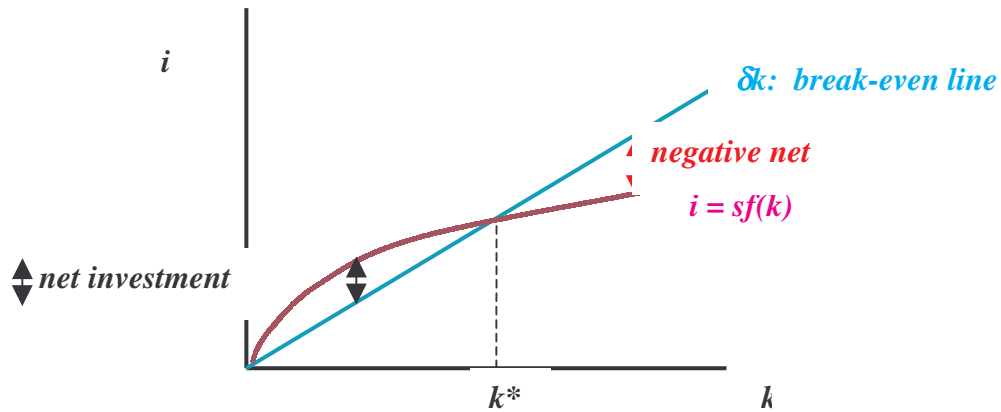
Figure 3.2 Investment and production functions



Thus, both output per worker and investment per worker are an increasing function (at a decreasing rate, because of diminishing MP_K) of capital per worker.

To show capital accumulation on the graph, we focus on the $i = sf(k)$ curve, and introduce depreciation.

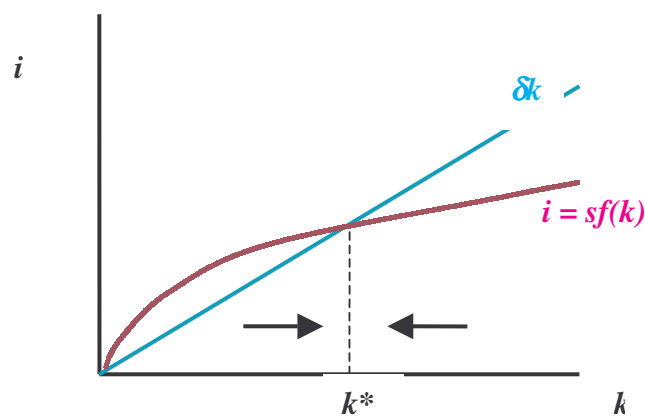
Figure 3.3 Investment and depreciation



Depreciation is a straight-line function of k . At some point, call it k^* , the depreciation line slices through the flattening investment curve. To the left of k^* , net investment is positive (gross greater than depreciation), to the right negative. Investment along the straight line just keeps capital worker constant, so we can regard the line as a *break-even investment schedule*.

In other words, to the left of k^* , k is increasing, to the right, k is decreasing. Therefore k^* is the *steady state* level of capital per worker—the long-run equilibrium of the economy.

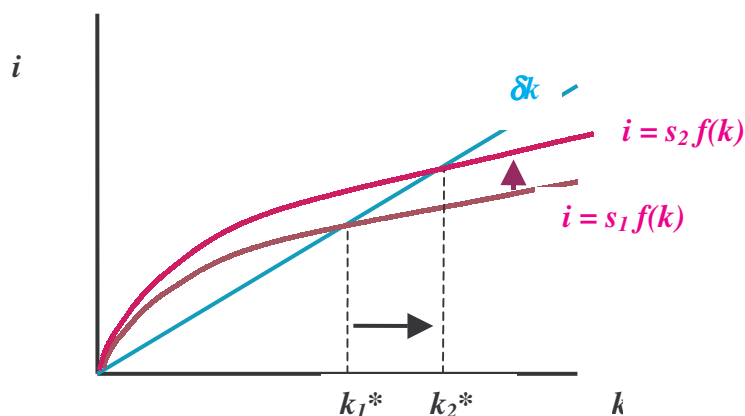
Figure 3.5: Steady-state equilibrium



Accumulation and growth

In equilibrium, with a given saving rate, there is no net accumulation of capital, and no growth of output. What if the saving rate goes up?

Figure 3.6 Transition to a higher-saving steady state



The capital stock rises eventually to a new steady state equilibrium, at k_2^* . During the transition output as well as capital grows, both at a diminishing rate. Growth tapers off to nothing in the new steady state.

Implications

A permanent increase in the saving ratio will raise the level of output permanently, but not its rate of growth. During the transition period, which might last decades, growth will be higher. But the increased investment eventually results in an offsetting increase in depreciation, and hence capital per worker levels off. Saving and capital accumulation on its own, with given technology, cannot explain long-run economic expansion.

Evidence

Current affairs

The US has a very large budget deficit, about 5% of GDP, for as far as the eye can see, with the Administration offering more tax cuts, and some high-priced new programs (medicine for seniors, Americans on Mars,). Budget deficits reduce national savings, which might not be too bad if the private saving rate were high, but in the US it is low. The Solow model warns that such a policy is likely to reduce income growth over an extended period.

3. Population growth

Accumulation to stand still

Population growth, of course, affects accumulation of capital per worker. To see how, start with the approximation that the proportional change in a ratio is equal to the proportional change in the numerator *minus* the proportional change in the denominator yields. The approximation is good for modest-sized changes, such as those for macroeconomic aggregates from year to year. Here the ratio of concern is $k = K/L$, for which the rate of change is:

$$\Delta k/k = \Delta K/K - \Delta L/L.$$

Since ΔK is equal to investment (I) minus depreciation (δK), rewrite this as:

$$\Delta k/k = I/K - \delta K/K - n.$$

For the change in the capital stock per worker, as opposed to the rate of change, multiply each side by k , or K/L , as convenient:

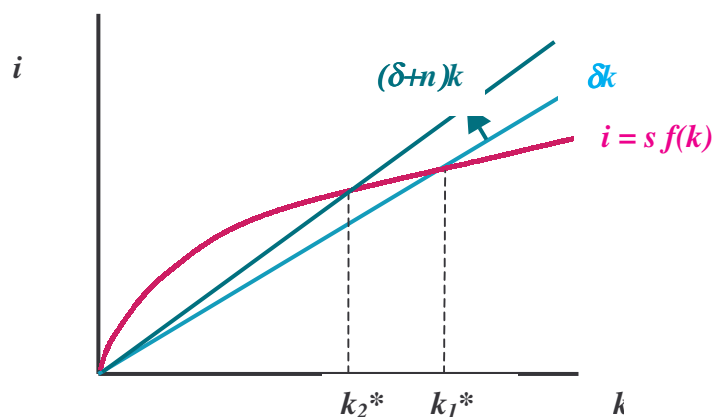
$$\Delta k = (I/K - \delta K/K)K/L - nk = I/L - \delta K/L - nk,$$

this simplifies to:

$$\Delta k = i - (\delta + n)k.$$

The change in capital per worker is given by net investment *less* the investment required to provide newly arriving workers with the same capital as existing workers, nk . The break-even line rotates upward.

Figure 3.7 Negative impact of population growth



Implications

Population growth, in itself, reduces the steady-state level of capital per worker. Via the production function, this translates directly to lower per capita output and income. Steady-state per capita income is constant; total output grows at the rate of population growth.

So far, the model does not explain permanently increasing per capita income (ironic, given the title of Ch 4)—for this we need improving productivity.

Evidence

What does the model explain?

Solow's model, even in a rudimentary version without technical change, explains

- positive correlation of investment rates and per capita income
- negative correlation of population growth and per capita income

It also helps explain these remarkable phenomena:

- 2-3 decade growth miracles following wartime destruction
- China and Asian tigers
- ultimate collapse of Soviet heavy industry expansion

4. Technological progress

Equilibrium with increasing productivity

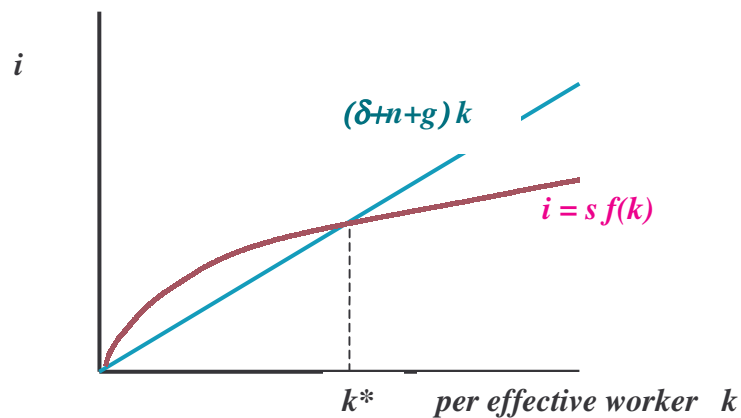
$$Y = F(K, L \times E)$$

We can measure labour in efficiency units, E . Technological progress (improved equipment, education, skills, health, infrastructure, etc.) increases the productivity of labour. Let this improvement be at a steady rate, g . Redefine k to stand for capital *per effective worker*, i.e. $k = K / (L \times E)$, and likewise $y = Y / (L \times E)$. The equation for capital deepening in terms of effective workers, Δk , is

$$\Delta k = i - (\delta + n + g)k.$$

The additional term gk represents investment that merely keeps the capital stock per effective worker constant as efficiency increases.

Figure 3.8: Equilibrium with technological progress



Capital per effective worker is in equilibrium at k^* , for the same reasons as in the constant technology case. An increase in g , just like an increase in δ or n , rotates the break-even line upward.

Capital per actual worker grows at rate g , as does output per worker (the capital/output ratio is therefore stable).

Table 3.1: Steady state with technological progress		
		Growth rate
Capital per effective worker	$k = K / (E \times L)$	0
Output per effective worker	$y = Y / (E \times L) = f(k)$	0
Capital & output per worker	Y/L & K/L	g
Total output	Y	$n + g$

Implication

Technological progress explains long-run expansion of income per capita.

Income and factor shares

The distribution of income between capital and labour remains constant along the steady-state growth path. The return on capital (in this model, the interest rate) is constant, while the stock grows at rate $n+g$. The wage rate grows at g , the labour force at n , so the wage bill also grows at $n+g$.

Table 3.2: Steady-state distribution of income		
	Golden Rule Level	Growth rate
Total income	Y	$n + g$
Return on capital (interest rate)	$MP_K = n + g$	0
Total return to capital	$MP_K K$	$n + g$
Wage rate	MP_L	g
Total return to labour	$MP_L L$	$n + g$
Capital share	$\alpha = MP_K K / Y$	0
Labour share	$1 - \alpha$	0

Evidence

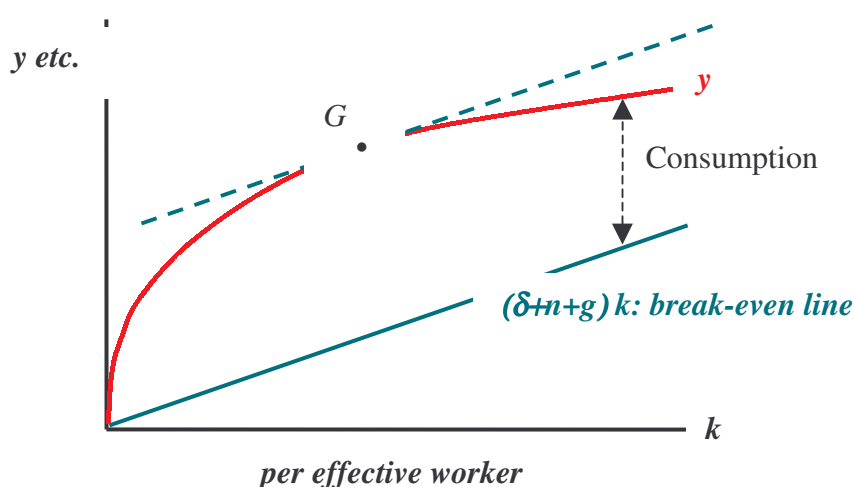
Factor shares have remained roughly stable, over long periods of time. In Canada and the US the labour share has been about 70%.

Golden rule

As we have seen, the equilibrium value of capital per effective worker increases with the saving ratio. In steady state, the per capita income path is higher for a greater savings ratio.

Is more saving always better? No. We want to maximize *consumption*, not *income*. A greater capital stock requires more break-even investment—i.e. investment just to keep capital per effective worker constant. At some point, because of decreasing MP_K , capital increases start to cannibalize capital itself, at the margin leaving nothing for consumption.

Figure 3.8: Consumption possibilities



To maximize consumption, we want to hold the capital ratio at the point where further increases in capital cease to yield a marginal gain in consumption. This point in Figure 3.8 is G . Beyond that point increases in capital per effective worker create *decreases in consumption*, despite continued increases in output.

The *Golden Rule* describes the consumption-maximizing path of capital accumulation corresponding to G . As a matter of geometry, at G the slope of the production function (the gross marginal product of capital) is equal to the slope of the break-even line. That is:

$$MP_K = \delta + n + g$$

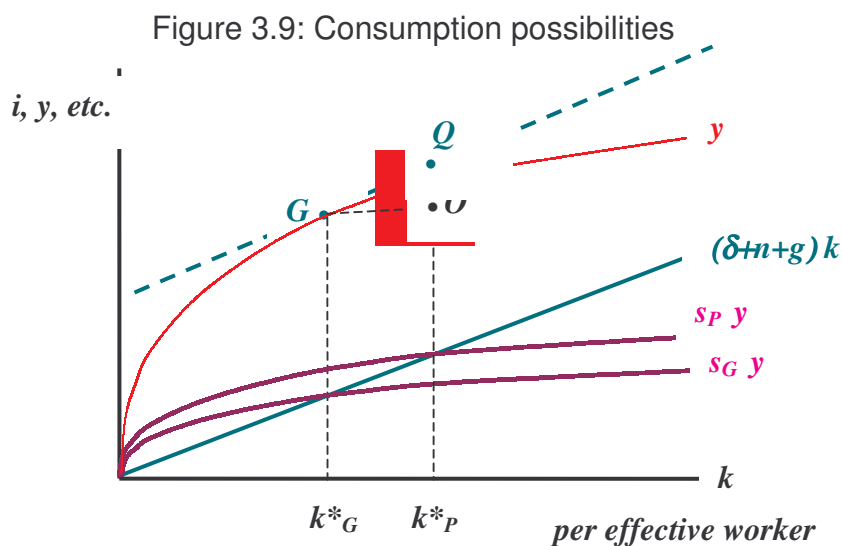
or

$$MP_K - \delta = n + g.$$

Under the Golden Rule, the net marginal product of capital is equal to the growth rate of total output.

Real interest and growth rates

In equilibrium, the interest rate (the return on saving) is equal to the net marginal product of capital after depreciation. If the interest rate is less than the growth rate, the economy is saving too much—i.e. consuming more provides a free lunch.



For example, at point P, the interest rate is less than the rate of growth. A reduction in the saving rate, from s_P to s_G , reduces steady state output (OP), but reduces break-even investment even more (OQ). [Magnify.]

An efficient economy therefore would have the long-run equilibrium real rate of interest (or *natural rate*) equal to, or above, the growth rate. The Golden Rule has equality.

To apply this to the real world one has to recognize that

- actual interest rates are many and varied, with different terms and risks and other features, depending on the asset and the borrower
- returns on capital may be highly uncertain, and therefore contain a risk premium over and above the interest rate on financial assets

Bearing in mind these qualifications, one can say that if the economy is operating efficiently, then in the long run the real yield on long-term government bonds (credit-risk-free assets) should be about equal, or slightly above, the growth rate.

Evidence

In Canada, over time, the average long-term bond yield has been in the same range as the rate of growth (3-4%). But the overall return on capital has been above this (M&S estimate 7 1/2%, p 11), reflecting risk premiums.

Figures showing time profile of consumption following initial change in saving ratio.