Solving for $\hat{\gamma}_1$ then yields

$$\hat{\gamma}_1 = \big(\boldsymbol{Y}_1^\top(\boldsymbol{M}_1 - \hat{\kappa}\boldsymbol{M}_X)\boldsymbol{Y}_1\big)^{-1}\boldsymbol{Y}_1^\top(\boldsymbol{M}_1 - \hat{\kappa}\boldsymbol{M}_X)\boldsymbol{y}.$$

Since $\boldsymbol{X}_1 \in \mathcal{S}(\boldsymbol{X})$, $\boldsymbol{M}_1 - \hat{\kappa}\boldsymbol{M}_X = \boldsymbol{M}_1(\mathbf{I} - \hat{\kappa}\boldsymbol{M}_X)$. Using this fact and a little algebra, which we leave as an exercise, it can be shown that $\hat{\gamma}_1$ can also be computed using the formula

$$\begin{bmatrix}\hat{\boldsymbol{\beta}}_1\\ \hat{\gamma}_1\end{bmatrix} = \begin{bmatrix}\boldsymbol{X}_1^\top\boldsymbol{X}_1 & \boldsymbol{X}_1^\top\boldsymbol{Y}_1\\ \boldsymbol{Y}_1^\top\boldsymbol{X}_1 & \boldsymbol{Y}_1^\top(\mathbf{I} - \hat{\kappa}\boldsymbol{M}_X)\boldsymbol{Y}_1\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{X}_1^\top\boldsymbol{y}\\ \boldsymbol{Y}_1^\top(\mathbf{I} - \hat{\kappa}\boldsymbol{M}_X)\boldsymbol{y}\end{bmatrix}, \tag{18.53}$$

which yields $\hat{\boldsymbol{\beta}}_1$ as well. Then if we define $\boldsymbol{Z}$ as $[\boldsymbol{X}_1 \quad \boldsymbol{Y}_1]$ and $\boldsymbol{\delta}$ as $[\boldsymbol{\beta}_1 \,\vdots\, \boldsymbol{\gamma}_1]$, as in (18.18), (18.53) can be written in the very simple form

$$\hat{\boldsymbol{\delta}} = \big(\boldsymbol{Z}^\top(\mathbf{I} - \hat{\kappa}\boldsymbol{M}_X)\boldsymbol{Z}\big)^{-1}\boldsymbol{Z}^\top(\mathbf{I} - \hat{\kappa}\boldsymbol{M}_X)\boldsymbol{y}. \tag{18.54}$$

Equation (18.53) is one way of writing LIML as a member of what is called the ***K*-class** of estimators; see Theil (1961) and Nagar (1959). Equation (18.54) is a simpler way of doing the same thing. The K-class consists of all estimators that can be written in either of these two forms, but with an arbitrary scalar K replacing $\hat{\kappa}$. We use K rather than the more traditional k to denote this scalar in order to avoid confusion with the number of exogenous variables in the system. The LIML estimator is thus a K-class estimator with $K = \hat{\kappa}$. Similarly, as is evident from (18.54), the 2SLS estimator is a K-class estimator with $K = 1$, and the OLS estimator is a K-class estimator with $K = 0$. Since $\hat{\kappa} = 1$ for a structural equation that is just identified, it follows immediately from (18.54) that the LIML and 2SLS estimators coincide in this special case.

It can be shown that K-class estimators are consistent whenever K tends to 1 asymptotically at a rate faster than $n^{-1/2}$; see Schmidt (1976), among others. Even though the consistency of LIML follows from general results for ML estimators, it is interesting to see how this result for the K-class applies to it. We have already seen that $n\log(\hat{\kappa})$ is the LR test statistic for the null hypothesis that the overidentifying restrictions on the structural equation being estimated are valid. If we Taylor expand the logarithm, we find that $n\log(\hat{\kappa}) \cong n(\hat{\kappa} - 1)$. Since this test statistic has an asymptotic χ^2 distribution, it must be $O(1)$, and so $\hat{\kappa} - 1$ must be $O(n^{-1})$. This then establishes the consistency of LIML.

There are many other K-class estimators. For example, Sawa (1973) has suggested a way of modifying the 2SLS estimator to reduce bias, and Fuller (1977) and Morimune (1978, 1983) have suggested modified versions of the LIML estimator. Fuller's estimator, which is the simplest of these, uses $K = \hat{\kappa} - \alpha/(n - k)$, where α is a positive constant that must be chosen by the investigator. One good choice is $\alpha = 1$, since it yields estimates that