where L denotes the **lag operator**. The lag operator L has the property that when L multiplies anything with a time subscript, this subscript is lagged one period. Thus

$$Lu_t = u_{t-1}, \quad L^2 u_t = u_{t-2}, \quad L^p u_t = u_{t-p},$$

and so on. The expression in parentheses in (10.36) is a polynomial in the lag operator L, with coefficients 1 and $-\rho_1, \ldots, -\rho_p$. If we define $A(L, \boldsymbol{\rho})$ as being equal to this polynomial, $\boldsymbol{\rho}$ representing the vector $[\rho_1 \vdots \rho_2 \vdots \cdots \vdots \rho_p]$, we can write (10.36) even more compactly as

$$A(L, \boldsymbol{\rho})u_t = \varepsilon_t, \quad \varepsilon_t \sim \text{IID}(0, \omega^2). \tag{10.37}$$

For the same reasons that we wish to impose the condition $|\rho_1| < 1$ on AR(1) processes so as to ensure that they are stationary, we would like to impose stationarity conditions on general AR(p) processes. The stationarity condition for such processes may be expressed in several ways; one of them is that all the roots of the polynomial equation in z,

$$A(z, \boldsymbol{\rho}) \equiv 1 - \rho_1 z - \rho_2 z^2 - \cdots - \rho_p z^p = 0 \tag{10.38}$$

must lie **outside the unit circle**, which simply means that all of the roots of (10.38) must be greater than 1 in absolute value. This condition can lead to quite complicated restrictions on $\boldsymbol{\rho}$ for general AR(p) processes.

It rarely makes sense to specify a high-order AR(p) process (i.e., one with p a large number) when trying to model the error terms associated with a regression model. The AR(2) process is much more flexible, but also much more complicated, than the AR(1) process; it is often all that is needed when the latter is too restrictive. The additional complexity of the AR(2) process is easily seen. For example, the variance of u_t, assuming stationarity, is

$$\sigma^2 = \frac{1 - \rho_2}{1 + \rho_2} \times \frac{\omega^2}{(1 - \rho_2)^2 - \rho_1^2},$$

which is substantially more complicated than the corresponding expression (10.02) for the AR(1) case, and stationarity now requires that three conditions hold:

$$\rho_1 + \rho_2 < 1; \quad \rho_2 - \rho_1 < 1; \quad \rho_2 > -1. \tag{10.39}$$

Conditions (10.39) define a **stationarity triangle**. This triangle has vertices at $(-2, -1)$, $(2, -1)$, and $(0, 1)$. Provided that the point (ρ_1, ρ_2) lies within the triangle, the AR(2) process will be stationary.

Autoregressive processes of order higher than 2 arise quite frequently with time-series data that exhibit seasonal variation. It is not uncommon, for example, for error terms in models estimated using quarterly data apparently to follow the **simple AR(4) process**

$$u_t = \rho_4 u_{t-4} + \varepsilon_t, \quad \varepsilon_t \sim \text{IID}(0, \omega^2), \tag{10.40}$$