and (18.20), be expressed as

$$\begin{aligned}\boldsymbol{\pi}_1 - \boldsymbol{\Pi}_{11}\boldsymbol{\gamma}_1 &= \boldsymbol{\beta}_1\\ \boldsymbol{\pi}_2 - \boldsymbol{\Pi}_{21}\boldsymbol{\gamma}_1 &= \mathbf{0}.\end{aligned}$$

The first of these two equations serves to define $\boldsymbol{\beta}_1$ in terms of $\boldsymbol{\Pi}$ and $\boldsymbol{\gamma}_1$, and allows us to see that $\boldsymbol{\beta}_1$ can be identified if $\boldsymbol{\gamma}_1$ can be. The second equation shows that $\boldsymbol{\gamma}_1$ is determined uniquely if and only if the submatrix $\boldsymbol{\Pi}_{21}$ has full column rank, that is, if the rank of the matrix is equal to the number of columns (see Appendix A). The submatrix $\boldsymbol{\Pi}_{21}$ has $k - k_1$ rows and g_1 columns. Therefore, if the order condition is satisfied, there are at least as many rows as columns. The condition for the identifiability of $\boldsymbol{\gamma}_1$, and so also of $\boldsymbol{\beta}_1$, is thus simply that the columns of $\boldsymbol{\Pi}_{21}$ in the DGP should be linearly independent.

It is instructive to show why this last condition is equivalent to the rank condition in terms of $\operatorname{plim}(n^{-1}\boldsymbol{Z}^\top\boldsymbol{P}_X\boldsymbol{Z})$. If, as we have tacitly assumed throughout this discussion, the exogenous variables $\boldsymbol{X}$ satisfy the condition that $\operatorname{plim}(n^{-1}\boldsymbol{X}^\top\boldsymbol{X})$ is positive definite, then $\operatorname{plim}(n^{-1}\boldsymbol{Z}^\top\boldsymbol{P}_X\boldsymbol{Z})$ can fail to have full rank only if $\operatorname{plim}(n^{-1}\boldsymbol{X}^\top\boldsymbol{Z})$ has rank less than $g_1 + k_1$, the number of columns of $\boldsymbol{Z}$. The probability limit of the matrix $n^{-1}\boldsymbol{X}^\top\boldsymbol{Z}$ follows from (18.22), with $\boldsymbol{X}$ replacing $\boldsymbol{W}$. If, for notational simplicity, we drop the probability limit and the factor of n^{-1}, which are not essential to the discussion, the matrix of interest can be written as

$$\begin{bmatrix}\boldsymbol{X}_1^\top\boldsymbol{X}_1 & \boldsymbol{X}_1^\top\boldsymbol{X}_1\boldsymbol{\Pi}_{11} + \boldsymbol{X}_1^\top\boldsymbol{X}_2\boldsymbol{\Pi}_{21}\\ \boldsymbol{X}_2^\top\boldsymbol{X}_1 & \boldsymbol{X}_2^\top\boldsymbol{X}_1\boldsymbol{\Pi}_{11} + \boldsymbol{X}_2^\top\boldsymbol{X}_2\boldsymbol{\Pi}_{21}\end{bmatrix}. \tag{18.23}$$

This matrix does not have full column rank of $g_1 + k_1$ if and only if there exists a nonzero $(g_1 + k_1)$–vector $\boldsymbol{\theta} \equiv [\boldsymbol{\theta}_1 \,\vdots\, \boldsymbol{\theta}_2]$ such that postmultiplying (18.23) by $\boldsymbol{\theta}$ gives zero. If we write this condition out and rearrange slightly, we obtain

$$\begin{bmatrix}\boldsymbol{X}_1^\top\boldsymbol{X}_1 & \boldsymbol{X}_1^\top\boldsymbol{X}_2\\ \boldsymbol{X}_2^\top\boldsymbol{X}_1 & \boldsymbol{X}_2^\top\boldsymbol{X}_2\end{bmatrix}\begin{bmatrix}\boldsymbol{\theta}_1 + \boldsymbol{\Pi}_{11}\boldsymbol{\theta}_2\\ \boldsymbol{\Pi}_{21}\boldsymbol{\theta}_2\end{bmatrix} = \mathbf{0}. \tag{18.24}$$

The first matrix on the left-hand side here is just $\boldsymbol{X}^\top\boldsymbol{X}$ and is therefore nonsingular. The condition reduces to the two vector equations

$$\boldsymbol{\theta}_1 + \boldsymbol{\Pi}_{11}\boldsymbol{\theta}_2 = \mathbf{0} \tag{18.25}$$

$$\boldsymbol{\Pi}_{21}\boldsymbol{\theta}_2 = \mathbf{0}. \tag{18.26}$$

If these equations hold for some nonzero $\boldsymbol{\theta}$, it is clear that $\boldsymbol{\theta}_2$ cannot be zero. Consequently, the second of these equations can hold only if $\boldsymbol{\Pi}_{21}$ has less than full column rank. It follows that if the rank condition in terms of $\boldsymbol{Z}^\top\boldsymbol{P}_X\boldsymbol{Z}$ does not hold, then it does not hold in terms of $\boldsymbol{\Pi}_{21}$ either. Conversely, suppose that (18.26) holds for some nonzero g_1–vector $\boldsymbol{\theta}_2$. Then $\boldsymbol{\Pi}_{21}$ does not have full column rank. Define $\boldsymbol{\theta}_1$ in terms of this $\boldsymbol{\theta}_2$ and $\boldsymbol{\Pi}$ by means