or, in more compact notation, as

$$\sigma_t^2 = \alpha + A(L, \boldsymbol{\gamma})u_t^2 + B(L, \boldsymbol{\delta})\sigma_t^2,$$

where $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ are parameter vectors with typical elements $\gamma_i$ and $\delta_j$, respectively, and $A(L, \boldsymbol{\gamma})$ and $B(L, \boldsymbol{\delta})$ are polynomials in the lag operator $L$. In the GARCH model, the conditional variance $\sigma_t^2$ depends on its own past values as well as on lagged values of $u_t^2$. This means that $\sigma_t^2$ effectively depends on all past values of $u_t^2$. In practice, a GARCH model with very few parameters often performs as well as an ARCH model with many parameters. In particular, one simple model that often works very well is the **GARCH(1, 1)** model,

$$\sigma_t^2 = \alpha + \gamma_1 u_{t-1}^2 + \delta_1 \sigma_{t-1}^2. \tag{16.21}$$

In practice, one must solve a GARCH model to eliminate the $\sigma_{t-j}^2$ terms from the right-hand side before one can estimate it. The problem is essentially the same as estimating a moving average model or an ARMA model with a moving average component; see Section 10.7. For example, the GARCH(1, 1) model (16.21) can be solved recursively to yield

$$\sigma_t^2 = \frac{\alpha}{1-\delta_1} + \gamma_1\big(u_{t-1}^2 + \delta_1 u_{t-2}^2 + \delta_1^2 u_{t-3}^2 + \delta_1^3 u_{t-4}^2 + \cdots\big). \tag{16.22}$$

Various assumptions can be made about the presample error terms. The simplest is to assume that they are zero, but it is more realistic to assume that they are equal to their unconditional expectation.

It is interesting to observe that, when $\gamma_1$ and $\delta_1$ are both near zero, the solved GARCH(1, 1) model (16.22) looks like an ARCH(1) model. Because of this, it turns out that an appropriate test for GARCH(1, 1) errors is simply to regress the squared residuals on a constant term and on the squared residuals lagged once. In general, an LM test against GARCH$(p, q)$ errors is the same as an LM test against ARCH$(\max(p, q))$ errors. These results are completely analogous to the results for testing against ARMA$(p, q)$ errors that we discussed in Section 10.8.

There are three principal ways to estimate regression models with ARCH and GARCH errors: feasible GLS, one-step efficient estimation, and maximum likelihood. In the simplest approach, which is feasible GLS, one first estimates the regression model by ordinary or nonlinear least squares, then uses the squared residuals to estimate the parameters of the ARCH or GARCH process, and finally uses weighted least squares to estimate the parameters of the regression function. This procedure can run into difficulties if the conditional variances predicted by the fitted ARCH process are not all positive, and various ad hoc methods may then be used to ensure that they are all positive.

The estimates of the ARCH parameters obtained by this sort of feasible GLS procedure will not be asymptotically efficient. Engle (1982b) therefore